Staircase inside leaning tower of Pisa, Italy
photo: Soňa Čeretková
STAIRCASE TO EVEN MORE INTERESTING MATHEMATICS TEACHING

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INTRODUCTION

Mathematics, its teaching and learning and mathematics teacher training is nowadays a widely discussed issue. Undoubtedly, future mathematics teachers need to acquire profound mathematics knowledge in various mathematics branches. Moreover, it is desirable to ensure that during their studies mathematics teacher trainees encounter such topics which offer opportunities to gain knowledge and skills of adapting mathematical issues for use of modern digital technologies. Hand in hand, it should allow teacher trainees to get the hang of mathematical modelling of real situations and to master mathematical representations of other sciences covered in the curriculum of school subjects.

This university textbook serves to present up-to-date, tried and tested links between mathematical knowledge and real life issues, and to provide its readers with interesting interdisciplinary joints between mathematics and statistics with geography, physics, and economics. Merging the knowledge and experience of mathematics as well as non-mathematics teacher training experts from three universities from three European countries, namely Norway, Iceland and Slovakia, has resulted in texts which may become not only a source of materials for active teaching, but also an apt illustration and an authentic argument of the usefulness and importance of mathematical knowledge in diverse practice and real life.

The textbook is primarily intended for future and present mathematics teachers, though its content might be attractive and beneficial for anyone with advanced mathematical literacy and skills in using dynamical mathematical software.

The authors of innovative educational texts append also additional information about the methodical approaches applied in adaptations of particular topics, and indicate possible and practical continuation of developing particular attributes of the topics. The textbook may thus serve as an inspiration for proposals of diploma theses or dissertations concerned with the use of digital technologies and interdisciplinary links between school subjects.

This university textbook is an outcome of project Improving Quality of Higher Education Based on Development of Multilateral Institutional Cooperation (https://www.ukf.sk/en/project-nis). Within the project duration the authors of the texts presented the issues covered in this textbook to students and academic community at the participating universities as a part of short-term mobility events, at the Summer School for PhD students, and at the Autumn Conference for PhD Students. Publishing the texts in form of a textbook makes the innovative mathematics teacher training institutionalized, and the textbook itself remains a durable project outcome and an evidence of interesting and beneficial act within international scientific cooperation concerning with the theory of mathematics education.

Nitra, March 2016

Soňa Čeretková
Best experiences with implementing student active methods: results and challenges, ideas and tasks
Introduction

The chapter is divided into two main parts:

The first part presents best experiences from a number of research and development activities during the years 2008 - 2015 which aimed to innovate and improve teaching by the use of what may be called student-active methods of teaching including the use of ICT. These best experiences take the form of research results, theoretical considerations, arguments and viewpoints. They were chosen with the aim to inspire the target group of this university textbook to improve the quality of education within our field, and grouped under the headlines: 1. IBL, modelling and multidisciplinarity, 2. Students’ creativity in problem solving and 3. Visualization with the use of ICT. For more profound studies, the interested reader must go to the original papers and articles which can be found in the reference list. Two perspectives on improvement of higher education by implementation of IBL direct the presentation of our experiences:

i. The content of the concept of IBL: The meaning of IBL is discussed with relations to modelling, problem solving and multidisciplinarity from the perspective of students’ learning. Use of ICT is integrated in some of the activities.

ii. The implementation of IBL: The potentials are discussed for a change of teaching, and for teachers’ professional development, towards more student-active teaching. In particular, the introduction and use of ICT in teaching and learning mathematics was researched in one of the projects.

The second part presents concrete teaching materials and ideas as a means to help and inspire teachers and students to initiate concrete projects and experiments aiming in the same direction. The materials present a variety of tasks within i) algebra, ii) functions and iii) modelling, with descriptions of the background and the learning goals. In some cases, experiences and reflections from the try out are displayed together with the tasks.

I. Experiences from research and development activities

1. IBL, modelling and multidisciplinary teaching

This paragraph summarizes parts of (Andresen 2013) in which the concept of IBL was studied and discussed like it was interpreted in the European FP7 projects PRIMAS and Fibonacci. (Andresen 2013) concludes that IBL in the two projects were determined by good teaching practice in relation to certain goals rather than by a strict description of concrete content or methods. Next, the paragraph outlines the characteristics of multidisciplinary teaching and its implementation in Danish upper secondary school based on the report in (Lindenskov and
Seemingly, IBL and multidisciplinary teaching have to some degree common goals – hence, experiences from our research into multidisciplinary teaching may be useful for implementation of IBL. The teaching experiment about chemical and mathematical modelling in (Andresen and Petersen 2011) is presented as one example of modelling in multidisciplinary teaching and the experiences from the research project are briefly outlined.

**Inquiry based learning in the PRIMAS and Fibonacci projects**

As a means to direct future approaches to teaching and learning, the EU seventh framework programme (FP7) has funded a number of projects focusing on development of models for support and dissemination of inquiry based mathematics teaching and learning. FP7 did not offer any definition of the term ‘inquiry based’. Consequently, the individual projects in FP7 had to give meaning to the term either based on novel ideas or by taking experiences from earlier projects of ‘good teaching practice’ as their starting point.

(Andresen 2013) deals with two examples, namely the EU-funded projects PRIMAS and Fibonacci which both aimed at dissemination of inquiry-based (science and) mathematics education in Europe. PRIMAS and Fibonacci had received funding from FP7 which means that it met the recommendations in the 'Rochard-report': Science Education Now - A renewed pedagogy for the future of Europe.

The PRIMAS project built on the modelling project LEMA (Learning and Education in and through Modelling and Applications) whereas the Fibonacci project supported and distributed results and experiences from the SINUS and the Pollen projects, amongst others. The shared goals of PRIMAS and Fibonacci were to disseminate results, experiences and teaching materials from previous projects concerning the development of inquiry based mathematics teaching and to support teachers' professional development within this area. (Andresen 2013) studied how meaning was ascribed to the term ‘inquiry based’ in the PRIMAS and the Fibonacci projects, under a bidirectional process of shaping and being shaped by practice. The study aimed to throw light on the question: How do the projects determine and stimulate 'inquiry' in school mathematics?

Central to the study in (Andresen 2013) was that the process of giving meaning to the concept of IBL and successively developing IBL, combined the change of norms and believes in the classroom about teaching and learning mathematics with general principles for teachers’ professional development. The study was founded theoretically on the combination of the interpretative framework explained in (Yackel & Rasmussen, 2002), with surveys of recent research as a background for considerations about the aspects of professional development for teachers summarized in (Sowder 2007).

**Inquiry based activities in the classroom**

In the study of PRIMAS and Fibonacci in (Andresen 2013), this framework offered an interpretation of ‘inquiry based’ activities in the classroom which is of common interest for professionals who wants to work with IBL: Following Yackel and Rasmussen (2002, p 316), ‘inquiry based’ is indicated by the social norms that students are expected to develop personally meaningful solutions, to explain and justify their thinking, to listen to and attempt to make sense of the thinking of others, and to raise questions and challenges when they disagree or do not understand. Part of this interpretation is also the socio-mathematical norm that explanations should be descriptions of actions on taken-as-shared mathematical objects that are experientially real for the students. The point is that when beliefs are seen
as the cognitive basis used by individuals to interpret situations arising in the course of social interaction, the evolution of the students’ beliefs over time may be studied by considering the reflexive relationship between beliefs and norms. The norms, then, are indicative of the students’ beliefs. In this interpretation the change towards ‘inquiry’ in the mathematics classroom will imply development of the appropriate social and socio-mathematical norms and of the students’ beliefs about what constitutes mathematical activity, about their own role and about the teacher’s role.

**Successful professional development for teachers**

The teachers who were in charge of this development of norms and beliefs were supported in both projects by website resources, courses, seminars and other activities for professional development. (Andresen 2013) studied the projects’ support to the participating teachers based on the article ‘The mathematical education and development of teachers’ in NCTM’s Second Handbook, where Judith T. Sowder addresses what it means to prepare teachers of mathematics and to support them, while they progress in their careers, with the professional learning opportunities they need to lead their students to succeed in learning mathematics (Sowder, 2007). The paragraph in (Sowder 2007) on the question: ‘What principles can be used to guide the design of professional development?’ served as a platform for the study of PRIMAS and Fibonacci in the perspective of professional development. It may be useful as a background for implementation of IBL in a broader context as well.

In that paragraph, Sowder compared three lists of elements of successful professional development described in Hawley and Valli (1999), Elmore (2002) and Clarke (1994), concluding that they have a great deal in common. The concluding list of elements is of general interest for implementation of IBL:

a. the role of determining the purpose of a professional development program,

b. the role of teachers in deciding on foci,

c. the need to have support from other constituencies to undertake changes in instruction,

d. the important role of collaborative problem solving,

e. the need for continuity over time,

f. the necessity of modelling the type of instruction expected,

g. the need for assessment that provides teachers with feedback they need to grow.

In addition, Sowder refers to a recent study (Garet, Porter, Desimone, Birman & Yoon, 2001) of professional development for more than a thousand mathematics and science teachers. The researchers in this study found that

h. sustained and intensive professional development was more likely to be effective, as reported by teachers, than was shorter professional development.

The results of the study also indicated that professional development was more likely to produce enhanced knowledge and skills, if it

i. focused on academic matter (content),
j. based on providing teachers opportunities for ‘hands-on’ work (active learning),

k. integrated into the daily life of the school (coherence)

Thus, the researchers in Garet et al. (2001) confirmed the importance of professional development that is focused on mathematics content, and their results support literature that said that coherence and collective participation were related to improvements in teacher knowledge and practice.

Reasons for the ineffectiveness of programs are also referred to by Sowder: the author indicates that teachers are likely to reject knowledge and skill requirements when

l. the requirements are imposed or encountered in the context of multiple, contradictory and overwhelming innovations,

m. teachers (except for those selected for design teams) are excluded from the development,

n. professional development is packaged in off-site courses or one-shot workshops that are alien to the purposes and contexts of teachers’ work,

o. teachers experience them alone and are afraid of being criticized by colleagues or of being seen as elevating themselves on pedestals above them.

Conclusions, (Andresen 2013)

The main conclusion of the study in (Andresen 2013) is summarized in the following paragraph. For the two projects’ descriptions of IBL, see: http://fibonacciproject.eu/ and http://www.primas-project.eu/en/index.do respectively. For the analysis and discussion of the research question, see (Andresen 2013).

The two FP7 projects’ guidelines, structures and dissemination models were designed to form a profile rather close to successful development projects according to recent research surveys. They built on earlier projects’ results and experiences with regard to good teaching practice which they, consequently, were very likely to successfully disseminate. By shaping the meaning of ‘inquiry’ from, and implementing these elements of good practice, the two projects had the potential to influence teaching practice in Europe at a large scale compared to local or even national development projects.

Initially, in both projects the meaning of ‘inquiry’ was only preliminarily characterized and the guidelines for professional development events and dissemination events did not give decisive criteria for what should be seen as ‘inquiry based’ mathematical activity and learning. A thought experiment was carried out in (Andresen 2013) to inquire this claim: Excerpts from an episode of students’ group-work was picked out from a research project neither focusing on inquiry nor the opposite. The students’ working style and approach as they appeared in the episode were interpreted in the thought-experiment with the aim to test whether it would be possible to assess or not, if they were engaged with ‘inquiry’ according to the two FP7 project’s meaning of the term. The thought experiment demonstrated that the projects’ construct would not be sufficient to discern a more superficial ‘inquiry’ behaviour from profound inquiries involving students’ mathematical conceptions and beliefs about mathematics. Consequently, there is a risk according to (Andresen 2013) that the projects’ structures and dissemination models might lead to the shaping, by practice, of a rather broad meaning of the term ‘inquiry’. An explicit focus would
be needed during the projects’ activities not only on the normative but also on the psychological perspectives and in particular on the students’ beliefs.

Correlations between modelling and multidisciplinary teaching

A new construct, ‘multi-disciplinarity’, was prescribed in Danish Upper Secondary Schools’ curriculum by governmental regulations in 2006. The Ministry’s intentions and requirements were centred on applications of and reflections upon each subject. The revision of mathematics teaching intended to support the students’ knowledge about ‘important aspects of the interplay between mathematics and culture, science and technology’. The students were also supposed to acquire knowledge about ‘how mathematics adds to understanding, formulating and treating problems in different subject areas’ and to know about mathematical reasoning. These learning goals served as a basis for the design of multidisciplinary mathematics teaching, which also intended to result in knowledge that enables the students to competently take a position on the applications of mathematics and to pass further education involving mathematics. Multi-disciplinary teaching was planned and carried out in projects where teams of teachers let the subjects collaborate. Apparently, the goals and means of multidisciplinary teaching has a lot in common with the goals and means of IBL. Experiences from research in cases of multidisciplinary teaching therefore may be fruitful for implementation of IBL.

Teachers and multidisciplinarity

According to (Andresen and Lindenskov 2009) multi-disciplinarity was defined in contrast to cross-disciplinarity, where the boarders between subjects are cancelled, and in contrast to trans-disciplinarity which does not acknowledge the division of knowledge into subjects. Hence, multi-disciplinary teaching gives opportunity to let the subjects’ mutually fertilise and, in parallel, to shape a clear picture in the students’ mind of the characteristics of the individual subjects. It seems that the didactical potentials of multi-disciplinary teaching resemble some well-known potential of inter-disciplinary activities. Apparently, the concept of multi-disciplinary teaching will fit better with mathematics teachers’ culture than is commonly the case for cross- or trans-disciplinary teaching, maybe because of the Danish structure of higher education: Upper secondary school teachers have to take their degree at masters’ level in two scientific subjects and then, afterwards, spend two years as a trainee to qualify for teaching. Because the educational courses are not an integrated part of the scientific study, most teachers would give priority to subject over didactics, rather than vice versa. In line with this, mathematics teachers tend to concentrate on the mathematical content also when collaborating with other subjects. Since teachers are required to qualify in two subjects, a number of mathematics teachers, though, are familiar with such collaborations. Common combinations with mathematics are, beside physics and chemistry, Danish language and societal sciences (Andresen and Lindenskov 2009).

Modelling chemical equilibrium in school mathematics with technology

This paragraph briefly gives an example of mathematics enrolled in multi-disciplinarity, which was tried out in Danish upper secondary school and reported in (Andresen and Petersen 2011). The topic was chosen as an example of intertwined mathematical modelling and chemical modelling, with the aim to realize a multidisciplinary teaching perspective on modelling in both subjects. The core issue of the case was to study the students’ perceptions of models and modelling, developed during a teaching sequence where the teacher deliberately focused on the intertwining of modelling concepts from mathematics and
chemistry. The aim of the study was to inquire multi-disciplinarity as a means to support students’ reflections about modelling and to support their modelling competencies in general.

In addition, the topic was suitable for study of the impact of technology use on students’ learning processes since it allowed to combine authenticity of both the chemical and the mathematical model with the potentials for fitting with approximations and use of technology, with elements of each of the four approaches mentioned in (Confrey and Maloney 2007 p 57):

1) Teach concepts and skills without computers and provide these technological tools as resources after mastery: i.e., solving the systems of equations by hand and subsequently introduce the discussion of different approximations and the use of MathCad,

2) Introduce technology to make patterns visible more readily, and to support mathematical concepts: in our case, the concept of the sets of solutions and how to choose between them was supported,

3) Teach new content necessitated by a technologically enhanced environment; in our case handling systems of eight or more equations and unknowns

4) Focus on applications, problem solving and modelling and use the technology as a tool for their solution: this was the very aim of the equilibrium project.

Data consisted of teaching materials (separated into a teacher’s part and a student’s part) prepared by A. Petersen (Petersen 2009) who taught the class chemistry, the students’ written reports, and notes from informal talks with A. Petersen. The teacher’s part of the teaching materials was of particular interest for our analysis because it articulated the intentions and reflections behind the choice of topics and the design of the complete teaching sequence. The students' part of the teaching materials gave information about the basis for their work, supplied by details from the informal interview with Mr. Petersen.

Conclusion

In spite of the good intentions, the excellent teamwork and the very promising subject, a number of weak points were registered in (Andresen and Petersen 2011) concerning the try out of the chemical-mathematical modelling project. For the detailed discussion, see (Andresen and Petersen 2011). These conclusions may also be relevant for IBL projects that resemble the multidisciplinary modelling project.

Technical obstacles

There was a lack of time for the students to prepare the written reports. The project lasted two week including one day in the chemistry lab. Consequently, most of the students spent too long time on the experiments and, subsequently, had to write the report without having time for profound discussions in their working groups or for substantial supervision from the teacher during the writing. The lack of time was reflected in the reports’ short, summary paragraphs on conclusions and perspectives which did not match the introductions’ presentations of aim and goals. The instruction sheets for the experiments were not tailored for this project. It was a common pedagogical practice in this class, that the students had to somehow modify or alter their working sheets before the experiments, to ensure that they do not only experience ‘cooking-book’ exercises in lab. For another try out, though, the
teacher would prefer to prepare new working sheets, tailored for this experiment. Measuring potentials of electrodes, which was part of the experiment, was a new method for the students. To reduce the complexity in a future try out, the students should be asked to make a small experiment using this method, before the equilibrium project runs.

Few students’ reflections

To encourage students’ reflections upon modelling and strengthen the written conclusions and perspectives, classroom reflection-discussions should be introduced in future repitions of the project, as a forerunner and support for the writing of (the last parts of) the reports. Such reflection-discussions could aim to balance out the students’ ‘technical-application’s view by explicit reflections upon the use of models and upon the modelling process. The request of explicit reflections as part of the written reports’ conclusions should ensure that more weight will be put on this important part.

Little focus on modelling

The project’s intention, to set chemical and mathematical modelling and the connection between the two in focus of attention was not really fulfilled in this first try out. One reason, apparently, may be the fact that the model of chemical equilibrium is based on fundamental principles like the law of mass action and the laws of conservation. Such fundamental principles are rarely discussed in the classroom; more often, they serve as a prerequisite embedded in the basis for treatment of their consequences in series of concrete or special cases. As a forerunner for the project, an example of a less fundamental and trusted scientific model could be a topic for one or two lessons, like for example theories about earth rays or phlogiston. The aim of including this forerunner in the project should be to make the students aware of scientific models’ role of giving explanations for observations, and to establish a shared basis for discussion of criteria for validity of such models. On this background, the possibility to compare different models of chemical equilibrium, resulting from different approximations, could be an issue of discussion also in future projects. The students should then be requested to make a comparison between at least two different models (with three, four or eight equations) in their reports.

Perspectives

This case study leads to draw an inference in line with the concept of forced autonomy introduced by Jeppe Skott in (Skott 2004). Like in Skott’s study, the new formal requests of multidisciplinary teaching leave the teacher in a situation, where ‘expected classroom practices and learning outcomes (are) formulated outside the classroom, but there is no set of well-defined methods for the teacher to carry out and only vague hints as to what kind of practice a certain situation may require. ‘. Skott argues in his study, that the notion of forced autonomy, based on the conceptions of mathematics and mathematical learning, should be extended to encompass not only the roles of the teacher when supporting students’ learning in classrooms, but also the multitude of other obligations that emerge in the course of the classroom interactions. In the case of multi-disciplinarity, the teacher’s situation appears even more complex when the perspectives from different subjects has to be connected or even intertwined in a challenging teaching task that involves theory as well as practical activities.

An extended notion of forced autonomy may, according to (Skott 2004), serve as a better means for researchers to understand the teachers role for the enacted curriculum. In our
case, the complexity of the multidisciplinary teaching sequence may serve to explain why the students' understanding of connections between mathematical modelling and chemical modelling, as it was revealed in their written reports, was rather loose in spite of the deliberate focusing.
2. Students creativity in problem solving

This paragraph reports from a research project referred to in (Andresen 2015a, 2015b, 2015c), titled 'Students’ strategies for problem solving'. Next, after the presentation of the project and some of its outcomes, the project is studied as a case of professional development. The aim of the project was to develop and study teaching that encourages students’ activity, inquiry and autonomy, and corresponding goal directed mathematics learning.

Background

The project was part of the EU project KeyCoMath (http://www.keycomath.eu/). It started in the spring 2013 where a collaborative research group was formed consisting of eight mathematics teachers from five local, upper secondary schools and one university researcher in mathematics education. The group had articulated certain concerns like:

   i. Students are too dependent of check lists and working habits; they seldom are able to ‘think outside the box’

   ii. Even the brightest students can reproduce, but rarely produce mathematical thinking

   iii. Many students do not want to solve new problems or to answer new questions.

The group had the hypothesis, that appropriate problem-solving environments could support realization of many students’ hidden potentials for independent, mathematical thinking. In accordance with the discussion of multidisciplinarity in a previous paragraph, independent, mathematical thinking can be related to one strand of what may be called inquiry based learning. We formulated the research question: “What strategies can we identify when the students work with problem solving in an inquiry based learning environment in upper secondary mathematics?” During the first year of the project, the teachers had designed and taught sequences in their own classes of about ten lessons each, where the students worked with problem solving. Each teaching experiment was observed and video recordings, field notes, students’ products and teaching materials were collected. Two interviews with the group, notes from work meetings and informal meetings were included in data. (Some of the materials in part II of this chapter were collected/created and tested in the project). Gradually, our group’s research interest concentrated on students’ modes of reasoning and, in particular, on ideas about mathematical creativity presented by Lithner (2008). The teachers deliberately designed sequences, which should provoke examples of mathematical creativity. During the following years, we studied these teaching experiments and analyzed data with the aim to study episodes of mathematical creativity. We designed and taught new teaching experiments based on experiences from the first round, and new data was collected similar to the first round.

Theoretical foundation for the project

The research group’s work was based on Polya’s problem solving heuristics (Polya, 1985), Alan Schoenfeld’s theories about mathematical thinking and problem solving and Johan Lithner’s research into students’ strategies for solving tasks and problems in mathematics (Lithner 2008). The research methodology was in line with a sociocultural perspective and encompassed collaborative teaching experiments (Cobb 1999). Data interpretation and analysis took norms and beliefs as its starting point and included social and psychological perspectives (Yackel and Rasmussen 2002). These theories and perspectives form
a foundation which would be suitable in general for research, studies and experiments with implementation of IBL in mathematics teaching.

During the project, we combined the teachers’ designs of materials, and their teaching experiments, with discussions and exchange of experiences in a number of joint meetings. When the idea of GMC (see below) emerged during the analysis of data, the concept of metarepresentational competence (diSessa 2002, 2004) was included as part of the research project’s theoretical framework.

**Polya and Schoenfeld: how to solve it and what it takes to solve it**

A central theoretical contribution of Alan Schoenfeld’s problem-solving research was his framework for analysis of mathematical problem-solving behavior. Based on discussions in the research group of (Schoenfeld, 2011), we decided to divide the teaching experiments into two separate parts. In every classroom experiment, the first session contained an introduction to an inquiry, problem solving working style. The other part was the main problem solving session. We planned to let the teaching of mathematical problem solving include explicit use of Polya’s scheme (Polya, 1985).

The teachers did not in advance see this as a major change of their classroom practices because they felt that problem-solving strategies would also be taught normally, although implicitly. However, they had the general impression that their students needed elementary problem solving tools like, for example, those strategies based on Polya’s scheme. The teachers wanted to enable students to make progress on their own hand rather than call for help as soon as they felt lost. In particular, some of the teachers also wanted to get rid of the students’ very close use of the textbook’s list of answers to the tasks. The experiment aimed to widen the students’ picture of mathematics in the direction of a subject open for ideas and including discussions based on mathematical knowledge and imaginations. The teachers wanted to change the students’ beliefs about mathematics and about their own roles, and the project intended to contribute to a change of the classroom’s norms and practices. These goals would not be far from those of an experiment which aimed to implement IBL.

The teachers felt comfortable with the preparation of materials for both parts of the teaching experiment, supported by discussions in the group and in smaller meetings.

**Lithner: Types of reasoning for solving tasks**

According to (Lithner, 2008), solving a task can be seen as carrying out four steps:

1) A (sub) task is met, which is denoted problematic situation if it is not obvious how to proceed.

2) A strategy choice is made. It can be supported by predictive argumentation: Why will the strategy solve the task?

3) The strategy is implemented, which can be supported by verificative argumentation: Why did the strategy solve the task?

4) A conclusion is obtained.

Further, Lithner discerns between different types of reasoning involving strategy choice and strategy implementation. The two main types of reasoning are IR (Imitative Reasoning) and CMR (Creative Mathematically founded Reasoning). IR encompasses i) memorised reasoning where the strategy choice is founded on recalling a complete answer and the strategy
implementation consists only of writing it down, and ii) three subtypes of algorithmic reasoning where the strategy choice is to recall a solution algorithm without creating a new solution; hereafter, the remaining parts of the strategy implementation are trivial.

In contrast, CMR fulfills all of the following criteria (Lithner, 2008) p 266:

1) Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created.

2) Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.

3) Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning.

Lithner did his studies at undergraduate level. Our group decided to take students’ CMR as a goal for the teaching experiment. Therefore, the data analysis concentrated on the identification of episodes of students’ creative mathematical thinking.

diSessa: metarepresentational competence

Metarepresentational competence (MRC) refers to the full complex of abilities to deal with representational issues. It includes, centrally, the ability to design new representations, including both creating representations and judging their adequacy for particular purposes. But it also includes understanding how presentations work, how to work presentations for different purposes and, indeed, what the purposes of representations are. Knowledge that allows students to learn new representations quickly and the ability to explain representations and their properties is also included (diSessa 2002). Representational literacy is important for the students’ critical capabilities (meaning the capability of judging the effectiveness of the design’s result, and of redesigning it) in MRC, according to (diSessa 2002). According to diSessa (2004), MRC may account for some parts of the competence to learn new concepts and to solve novel problems. Our group’s observations were in line with this and gave inspiration to new inquiries. Further, diSessa (2004) suggests that because insight and competence often involve coming up with an appropriate representation, learning may implicate developing one’s own personally effective representations for dealing with a conceptual domain. Although these two studies (diSessa 2002, 2004), in contrast with our project, aimed at linking metarepresentational competence with design, and with students’ critical capabilities, we found the concept of metarepresentational competence potentially useful for analysis of the GMC’s occurring from our data. In our project the students worked with different representations like formal expressions, graphs and pictures and in some of the cases they also used calculator, excel spreadsheet and GeoGebra. The linking of, and change between different representations of the same subject or entity were crucial steps towards solving the problem in a number of cases.

Glimpses of mathematical creativity (GMC).

One important result of the project was a didactic concept, which emerged during the analysis of data from the teaching experiments: a particular aspect of the students’ work, which we interpreted as glimpses of mathematical creativity (GMC). The GMCs occurred
when pairs of students worked together in the experimental learning environment. Characteristic of the environment was the demand, that the students engaged in solving a mathematical problem which was completely new to them, and also that the teacher deliberately would avoid to interfere by, for example, asking the students sub questions or structuring their actual process of problem solving. IBL should, preferably, be thought in a similar setting.

Analysis of data from the project had revealed several episodes of creativity, see for example (Andresen 2015a, 2015b). The discussion in (Andresen 2015a) of an episode from data concluded that the episode contained an example of the algorithmic reasoning (AR) described by Lithner (2008), and it also contained examples of initial creative thinking. All the episodes were examples of creative thinking concerning elementary reasoning rather than problem solving but they took place in a problem solving setting. In parallel, the discussion in (Andresen 2015b) compared another episode with Lithner’s criteria for CMR (Lithner 2008): The four criteria for CMR were fulfilled in the case, only if shifts between different representations can be seen as part of the ‘mathematical foundation’ in the third one. The GMC in the episode was founded on the students’ competence in shifting between the different representations (numbers, formulas and drawings) of the polygonal numbers. Their arguments for supporting the strategy choice were anchored in both students’ representational literacy, which is an aspect of metarepresentational competence (MRC) as it was described above.

One example of the students’ representational literacy was revealed in the episode, where one of the students talked about the formula for the triangle and for the square, and about adding these two, without even to discern between the different representations. The other student immediately understood the idea. This episode illustrates how an experimental lesson on polygonal numbers, founded on interplay between different representations, could be supportive of the students’ development of metarepresentational competence as well as their creative, mathematically founded reasoning.

The student’s creative reasoning in both episodes happened in a glimpse. In one episode, it happened when the students E1 and E2 realised that they had squares and square numbers. In the other episode, it happened when the student caught the connection between the pentagon consisting of a triangle (with a corresponding formula) and a square (with a corresponding formula) on the one hand, and, on the other hand, the algebraic number of which he wanted to have a formula. Hence, in both episodes the GMC happened in the moment, when the student managed to see the two as different representations of the same object. A number of episodes from our data contains examples of GMC, which happen in a similar way when a student manage to establish a link between two different representations of the same object. Further analysis of data from our group’s experiments may provide interesting insight into the connections and relations between CMR, GMC and MRC.

**The project as a case of professional development**

To give an impression of the teaching experiments’ content, four of the problem solving tasks are briefly summarised in the following:

*To model the movement of one gondola in a Ferris wheel (Two teachers, in two classes)*
Task: Make a mathematical description of the movement of one gondola in the Ferris wheel in Liseberg, Göteborg.

Pedagogical pointe: The students are not familiar with the mathematical descriptions of a harmonic oscillator. They learn to use the program GeoGebra, which helps them link the graph of the movement to the equation describing it. When they use the computer program they can focus on the problem at hand, and quickly and simply try out ideas and guesses to find the solution.

To renovate and furnish an apartment (One teacher in one class)

Task: Make a budget based on an amount of money and a floor plan of the apartment. Find out how much paint and tapestry will be needed, which furniture will fit in and can be afforded, and make an assessment of how the apartment should be heated.

Pedagogical pointe: To put the students in a more realistic situation, where it is not clear from the beginning, which part of their mathematical skills will be useful. This is in contrast to the tasks in their textbook which are normally divided into subjects as triangles, equations, straight lines etc.

To study consumer behaviour (Two teachers, in two merged classes)

Task: Create a questionnaire with statistics about the subjects’ shopping habits, Christmas traditions, work out habits, and whether they go to the cinema, concerts, the theatre etc. Give a statistical presentation of data

Pedagogical pointe: The students learn to utilize simple statistic methods. Some of them asked random passers-by in the shopping centre Lagunen, whereas other groups made an online questionnaire and got a larger data set to analyse with statistics. As an added bonus, they also got new and interesting results.

To gain knowledge about mathematical functions via sketches and geometric figures (Three teachers, in three classes)

Task: Look at a sketch and understand which mathematical methods can be used to describe what is seen. Exercises in visual thinking picked out from (Nelsen 1993 and 2000) like for example: Explain this (printed) geometrical proof of the formula \[(a+b)^2+(a-b)^2=2(a^2+b^2).\] (Nelsen 1993 p20)

Pedagogical pointe: There is not one definitive approach to the assignment, therefore the students had to develop their own way to solve the problems in the groups. They got insight into a new approach or mathematical method and a deeper understanding.

All the eight experiments made use of one of these tasks. The second round’s experiment made use of similar tasks. The two interviews with the teachers served as the main basis for the project studied as a case of professional development.

Two interviews focusing on the realisation of the teachers’ visions

Five of the eight teachers had carried out their first teaching experiment when we made the first group interview with all eight teachers 24. October 2013. All eight teachers participated in the group’s meetings in spring 2014 and in the interview May 2014 after second round.

Both interviews with the eight teachers followed interview guides prepared in advance with the aim to find out how and to what degree the teachers’ visions and ideas had been realised.
during the project. The first interview guide included questions about differences between their normal teaching and the teaching experiment, advantages and obstacles and general remarks. Besides, they were asked about the further proceeding of the project, the apparent learning outcome for the students, themselves and for research. The second interview guide included more detailed questions about what had actually happened during the experiments concerning the students’ thinking, their learning outcome, the teaching materials and feedback. Besides, the teachers were asked to give more general comments on the entire project’s outcomes.

a) The teachers who created and used the Ferris wheel materials had prepared a lesson on Polya’s scheme and problem solving. They taught so-called science classes with highly motivated, bright students. These students were very engaged also in the problem solving teaching experiment and the larger part reached to appropriate conclusions and models of harmonic oscillations. One of these teachers expressed (in the second interview) that the students did frequently ask the questions ‘What do we know’ and ‘What are we supposed to find’ stressed in Polya’s scheme, as a sign of small change. One of those classes had unexpectedly good results in a test with little more open ended questions in trigonometry, performed shortly after the first experiment. One of these teachers suggested that the students had to start problem solving early in school if they should be able to develop such skills. The other teacher found that the teachers had to spend long time working in this style to embed it as a natural behaviour. Both teachers participated in the second round with new experiments in algebra. They felt very comfortable with second round’s teaching and made some adjustments compared to first round, for example, one of the classes’ students were guided rather closely through the second experiment’s tasks.

b) The teacher who created and used this task started with an open ended ‘warm up’ and a short introduction to Polya’s scheme. The class was beyond average and the students engaged in the experiments and performed well. They enjoyed the lessons. The teacher was not able to identify any signs of change after the experiment and did not have time to continue in the second round. This teacher found that the task appeared to be too little challenging with regard to mathematics. Consequently, the students were not forced to activate all their mathematical knowledge for problem solving. He stressed the importance of the students’ very strong expectations concerning school mathematics as an obstacle for sudden changes.

c) The two teachers who created and used this task did the experiments in joint classes (40 students in all). The students were introduced to statistics in one half day session with small creative games and open ended tasks. Both teachers were a little surprised after the first round because they were the ones to ask all the questions, not the students. They felt that they had to give a lot of help to the students in the form of these questions. They both participated in the second round with a project on modelling authentic data with the help of regression in GeoGebra. One teacher told that the brightest students did reach a point during the second round where they were able to ask critical questions, but the less bright did not. One of the teachers told in the second interview that the students had learned something but not what they were expected to learn. He found that they had seen mathematics as a tool for real world management of facts during the experiment in statistics. The
other teacher was convinced that he would continue in the same direction but he had to spend more time and get more used to the style.

d) The three teachers who created and used this teaching sequence in algebra and series made a short introduction to open ended tasks during students’ group work on tasks. The main experiment was based on tasks from (Nelsen 1993 and 2000). Two of the teachers reported that the students had engaged in the group work and enjoyed the sessions but they were not convinced that this would be of relevance for the written examination. The third teacher told that she had had the same reaction in her class but she had found a task from written examination last year almost like the ones they had worked with. This argument was accepted in the class. One of the teachers had felt a little lost in the project because she did not really see what she was supposed to do, another told that she had the feeling of doing very little new. Two of the three participated in the second round with an experiment on probability and differential equations, respectively. The third teacher did not have the time to do an experiment in second round but she was engaged in the project’s meetings etc. None of these teachers had recognised a change in the classrooms in the intended direction. One of them expressed that she might not have changed the classroom but her students were better prepared for the written examination now.

All the eight teachers had positive feedback from their students, no one reported about resistance or negative response except the worry about the relevance for written examination mentioned above. The general impression was that the teachers had renewed their teaching in the desired direction but there was little effect, if any.

Results and conclusions

The overall conclusion was that the research project had served as a means for the teachers to get experiences with teaching according to their own desire with respect to students’ inquiry and intellectual independence in problem solving.

The detailed analysis lead to more diverse results. On the one hand the teachers were engaged with the project and all of them wanted to continue the project. They took advantage of the opportunity to try out experiments which were meant to realise their own ideas and visions. The teaching was planned, inspired by discussions in the group and based on materials created by them. Hence, all the teachers had a high degree of ownership to their actual teaching experiments. On the other hand, in spite of the group’s very good spirit there were elements of disappointment in the teachers’ reports in the second interview. The teachers had had the feeling of doing something differently in the classroom without to see clear consequences. The lack of the desired results was explained in the second interview mainly with reference to i) the period of time spent on the experiments compared to the students’ differing expectations and ii) students’ earlier experiences with school mathematics.

In terms of Cobb et al.’s framework one could say that the teachers’ beliefs about mathematics and beliefs about mathematics teaching were enacted in the design and completing of the experiments. The teachers’ shared visions or beliefs gave rise to a need for establishing and developing corresponding norms, not only amongst the teachers in the group but in the classrooms as well. The experiments were not in conflict with the existing social norms in the classrooms, even if some of them might be ‘on the border’, but the classrooms’ norms did not correspond exactly to the experiments, neither. The experiments
aimed to challenge the sociomathematical norms in the classrooms but these did undergo very small changes, if any, during the project. As a consequence, the students’ beliefs would probably stay almost untouched.

**Perspectives**

Neither beliefs nor norms are established in the classroom and developed over night! This study of a research project seen as a means for professional development has shown that on the one hand, such a project can be suitable to frame teachers’ development and enacting of their visions and beliefs about mathematics and mathematics teaching. On the other hand, there is no guarantee that the desired goals and learning outcome for the students will be reached. One possibility for improvement of the process would be a prolongation in time of the project. Actually, we will add one or two more rounds of teaching experiments in line with the first two. The inertia in changing the classroom’s norms and, intertwined with these also the beliefs, is a more profound issue. According to my view, the research perspective of the project must come into play here. When the analysis of data has been completed the results in the form of knowledge about the students’ strategies in problem solving activities will serve as guidelines for teaching. The teachers will be empowered with insight into new tools and resources for the students, and integrate these in their introductions to problem solving in the next teaching experiments. We shall see whether or not the teachers, then, will have a chance to supply the students with tools strong enough to start the process of change.
3. Visualization with the use of ICT

This paragraph about the use of ICT in mathematics teaching reports from a study of an ICT-project titled ViT (Visualization of Transformations) which took place at University of Bergen. The aim of the ViT project was to develop and test an interactive tool for linear algebra and publish it on the project’s website: http://kurs.uib.no/vit/ovelser.html . The study reported in (Andresen 2014) explored the teachers’ implementation of the project from a sociocultural perspective. The study’s research question was: What driving forces and what obstacles can be identified for implementation of a concrete change towards integration of visualization tools in a linear algebra course? Data for the study included interviews with the teachers and the rest of the project group, students’ answers to two questionnaires and tasks from the linear algebra course.

In this presentation, the ViT project and the study of it is related to a framework for interpretation and analysis of teachers’ professional development towards integration of ICT, developed a few years before in the Danish GeoGebra Institute when we designed inservice courses for mathematics teachers.

When teachers start teaching with the use of ICT

In (Andresen and Misfeldt 2011) the field of ICT and mathematics education was mapped into a small set of essential issues. The goal was to create a simple conceptual model which, in the design of courses for teachers offered by the Danish GeoGebra Institute, could serve as the foundation for two kinds of efforts: i) to combine mathematics education theory and practice, and ii) to develop a language to facilitate teachers’ and researchers’ discussion of the use of ICT in mathematics education, from a concrete as well as more theoretical point of view. Based on (Niss et al. 2002) and principles and standards (NCTM 2000) the following four essentials were pointed out, which we perceived as characteristic of and necessary for teachers’ professional development related to ICT integration in mathematics according to the authors:

1. Tool
2. Medium
3. Vehicle for learning
4. Change agent, encompassing:
   (a) Rethinking of teaching mathematics
   (b) New perspectives
   (c) New content

The essentials are briefly explained and their relevance motivated in the following. A more detailed discussion can be found in (Andresen and Misfeldt 2011).

Pointing out the essentials

In accordance with the new requests of competence the teacher is obliged to establish a learning environment where students can acquire competence to use appropriate, available ICT tools for mathematical learning purposes including task- and problem solving and modelling. To fulfil this, obviously, the teacher must be able to use the tool for problem solving and modelling him/her self. Work and study within this area is included in the term Tool essential.
The student is also supposed to develop competence to judge and make decisions about use of ICT tools, and to document his or her use. To be in front of this, the teacher necessarily has to work and study ICT tool from the perspective of a medium, meaning to deal with questions like what happens with mathematics during the transformation into and out from for example the laptop, how could the mathematical meaning be expressed and interpreted in different media, what could a semiotic view add to understanding etc. These issues are included in the term Medium essential.

To initiate, guide and support the students’ tool use for mathematical content learning purposes, the teacher has to work and study ICT tools in mathematics at a meta-level with regard to mathematical learning, which is termed Vehicle for learning essential.

The fourth, termed Change agent essential, is divided into subsections which may partly overlap the first three essential. Leaning upon the notion of teachers’ forced autonomy, such a Change agent essential can be interpreted as a characteristic, important for the teachers’ professional development. The teacher has to find a way to ensure that the students acquire these new competencies, and to evaluate the process. Like in Skott’s discussion of the forced autonomy of mathematics teachers (Skott 2004), this is a case of ‘expected classroom practices and learning outcomes formulated outside the classroom, but there is no set of well-defined methods for the teacher to carry out and only vague hints as to what kind of practice a certain situation may require’. Skott argues, based on his study, that the notion of forced autonomy, based on the conceptions of mathematics and mathematical learning, should be extended to encompass not only the roles of the teacher when supporting students’ learning in classrooms, but also the multitude of other obligations that emerge in the course of the classroom interactions, which further complicate teacher decision-making.

Danish studies of laptop-, computer- and handheld CAS calculator- classes (Andresen 2006) point to a similar complexity of forced, extra obligations, related to different aspects of the introduction of ICT tools into the classroom. Some additional extra obligations, though, did not relate to the introduction of ICT tools. Teamwork and interdisciplinary projects which were introduced by formal request simultaneously with the ICT in upper secondary constitute partly the context.

**The ViT project**

**Background**

The department of Mathematics at University of Bergen offers a second year course in linear algebra for around 250 students from different programs in mathematics and science such as pure and applied mathematics, physics, geophysics, statistics etc. The course was previously taught almost completely in algebraic representation and in general, many students found it difficult and/or tedious, as well as difficult to cope with. The ViT project, running from 2009 to 2013 and funded by 'Norges Universitetet' aimed to improve the linear algebra course in collaboration with its lecturers and the groups of student assistant teachers (groups of 3 - 6 persons each year). The ViT project was meant to offer the students an opportunity to widen their view on linear algebra leading to deeper understanding of the involved mathematical concepts and relations. The project’s means was to develop a visualization tool and to support the lecturers’ and the students’ use of the tool during the lectures. The visualization tool was created as a learning resource in the form of a web site with clusters of interactive sites (workshops), dealing with Vector spaces, Geometric Objects, Simple Transformations, and Determinants etc. (See http://kurs.uib.no/vit/).
Our study of the ViT project took place only during the last semester of the ViT project (spring 2012). It was based on data consisting of tasks and other teaching materials, evaluation sheets and interviews with the lecturers, the assistant teachers and with members of the ViT project group (5 interviews in all). Originally our intention was to study how the potentials of visualisations were realised in the ViT project, by observing students using the active sites while working with their tasks. It was our hope that such qualitative data would provide sufficient basis for an interpretation of the role of the sites as a resource for learning and thus the study would demonstrate the potential of visualisation-tools in a linear algebra course in this particular context.

However, a preliminary evaluation of the ViT project was carried out in 2011 through a students’ questionnaire with a much lower response rate than expected. The evaluation showed that the students had little interest for the visualisations. This result corresponded with the lecturers’ and the assistant teachers’ impressions from all three rounds of the ViT. Considering the fact that very few students actually used the sites, we found that it would be too difficult to carry out the study in accordance with the original plan. In contrast to the students’ apparent lack of interest, the lecturers and the ViT project group had found the idea of visualization of concepts and relations in linear algebra and the idea of interactive sites to support and deepen the students’ learning constructive and promising. The discrepancy between the mathematicians’ and the researchers’ views on the one hand, and the students’ lack of interest and lack of efforts on the other hand, caused a reformulation of the research question for our study of the project: What obstacles and what driving forces can be identified for implementation of a concrete change towards integration of visualization tools in a linear algebra course?

Use of technology

The introduction of technology use was a crucial element of the ViT project. Our study of the ViT project took as its starting point that in themselves, the interactive sites do nothing. According to the theory of Instrumental genesis and instrumented techniques, an artefact like for example the ViT sites do not in itself serve as a tool for anybody. It becomes useful, and then denoted an instrument, only after the user’s formation of one or more mental utilisation schemes. Such utilisation schemes connect the artefact with the user’s conceptual knowledge and with his or her understanding of the way it may be used to solve a given task. Thereby, the utilisation schemes contribute to the formation of instrumented action schemes. An instrument then consists of: i) the tool, for example the interactive sites, ii) the user’s mental utilisation schemes and iii) the task or problem to be solved. (Drijvers 2003). The term instrumental genesis denotes the process in which the artefact becomes an instrument (Drijvers et al. 2005). The formation of utilisation schemes and construction of instrumented action schemes proceed through activities in “The two-sided relationship between tool and learner as a process in which the tool in a manner of speaking shapes the thinking of the learner, but also is shaped by his thinking.” (Drijvers et al. 2005). In other words, in the ViT project the ViT sites would support for the students’ learning only if the students managed to include them, mentally, into their strategies, routines and resources for problem solving. The challenge for the teachers, hence, was to create a learning environment where the ViT sites served to form a technical dimension in the students’ mathematical activity, and to ensure that the activity would lead to useful visualisations of concepts and relations.
Two different approaches to the introduction of ICT tools

(Andresen and Misfeldt 2009) discerned between two different approaches to the introduction of computer based tool into the classroom. The study in (Andresen and Misfeldt 2009) built on observations of students’ resistance against development of personal tools from ICT artefacts and their teachers’ attempts to push this process. According to the theory of instrumental genesis the student must change his or her scheme of the mathematical problem situation including the mathematical concepts involved when developing a personal instrument. Hence, the students’ resistance in the study of the ViT project might be explained by noticing that apart from being intellectually challenging, the process of creating personal instruments may also challenge one’s perception of what mathematics is, i.e. the personal beliefs and norms. In order to implement ICT as part of the mathematical activities in the classroom the teachers will have to persistently insist on the students’ use of it. Two different approaches to the introduction of a new tool, were identified in (Andresen and Misfeldt 200) and represented by teacher A and teacher B:

i. Teacher A allowed the new technology to be included in the class’ problem work in a natural way. Not surprisingly, the appropriate use of this technology was rather sparse in the first lessons. Teacher A simulated an adidactical situation with regard to use of the tool, in the sense that focus was on strategies for solving tasks, on organising the group work and on the mathematical content, i.e. on calculus. Technology might be regarded simply as a part of the environment in teacher A’s class. It was one option and solution strategy that the students might choose.

ii. Teacher B, on the contrary, spent the first lessons with the new tool helping the students getting familiar with it. In teacher B’s class the introduction was a didactical situation, meaning that the students were asked to work with and learn about the tool, and trust the teacher’s claim that it would pay off in the long run.

According to (Andresen and Misfeldt 2009) three factors would, in general, influence teachers’ choice between A and B:

- The extent of the expected use of the tool. Teacher A expected the students to gradually progress in the process of instrumental genesis at his/her own pace in a self-regulating manner, ending up with incidents of tool use. In contrast, teacher B expected the students to fully integrate the tool in all their work. Teacher B thus, found that developing skills on their own would take too much of a bite out of the students’ intellectual capacity in the actual setting.

- The expected emergence of students’ motivation. This raised the question whether it was the teacher’s obligation to motivate the students for learning to use the tool, or if the existing social norms of the classroom would gradually develop and ‘feed’ the students’ motivation.

- The complexity of forced extra obligations related to different aspects of the introduction of ICT tools into the classroom. For example, technical obstacles and problems with hardware and software may force the teacher to change his or her teaching plans before or during the lessons. Teaching plans where the lessons are dependent of the tool seem to be much more fragile than plans where the tool is included ‘on demand’.
These factors were highly relevant for our study of the driving forces and obstacles related to technology use in the ViT project. For the complete discussion of this perspective, see (Andresen 2014).

Study of the ViT project

Our study of the ViT project encompassed two components:

• A background study into the goals and aims of the ViT project, required because the project had been running for two years at the time of our study. The background study inquired questions like: what norms and beliefs are targeted about visualisation and about the teaching and learning of linear algebra, and how and by whom did the project intend to develop them? The study also included the content of the website, for example testing whether it seemed to be of any help for non-experts of linear algebra.

• An evaluation of the ViT part of the linear algebra course based on interviews with lecturers and assistant teachers and on students’ questionnaires in 2011 and 2012

Data for analysis in the two parts was not completely separated. An interview with one of the lecturers for example, gave information about the aims and motives for the project but also about the insight from it.

Individual interviews with the 3 lecturers, group interview with 4 student teacher assistants and pair interview with two members from the ViT project group were carried out. The interviews were intended to provide i) concrete information about the ViT project, and insight into the participants’ view on ii) the learning potentials of visualization in linear algebra, iii) the possibilities for changing university education towards this direction, and iv) the students’ interests, abilities and motivation with regard to the ViT project.

Two questionnaires were distributed to the students in 2011 and 2012. The questions focused on the students’ impression of the websites and their role in the course, and on the students’ view on the website’s usefulness. The first questionnaire (in 2011) was handed out at the course’s last lecture, whereas the second questionnaire (in 2012) was handed out in connection with compulsory submission tasks. Even if the respondent rates to both questionnaires were too low to obtain significant results, answers to the second questionnaire inspired and informed the interview with the student teacher-assistants.

The textbook (excerpts), homework tasks, course plans and other teaching materials from the course served to illustrate the intended visualizations (see http://kurs.uib.no/vit/). Amongst these were specific tasks from the ViT project questioning visualizations and with reference to particular sites, created with the aim to encourage the students’ use of the sites. These tasks were included in the compulsory submissions from the students and as such, they served in our study to exemplify the lecturers’ attempts to guide the students.

Results

In general, the ViT project fulfilled important criteria for success summarised in (Sowder 2007). There was a high degree of alignment of content of the textbook, exercises, lectures and classroom teaching: the textbook claimed to give strong emphasis on geometric interpretation of every major concept. The lecturers had stressed the importance of visualisation too, with and without computer, (visualisation completely without computer, though, would be an option of no interest for this study of the ViT project). This was in line
with the ViT project’s ideas and aims which correlates strongly with the change agent essentials, and the mathematical content was coherent. The collaboration between the project group, the lecturers and the assistant teachers was good and there was a good atmosphere during the lectures and the classroom teaching.

Our study of the ViT project showed that the main barrier or obstacle was the computer use (for complete discussion, see Andresen 2014). The obstacles appeared at different levels and in different perspectives:

- Lack of alignment between the idea of implementing computer based visualisations into the course, and the requested hand-writing at the written examination. Request of documentation of use of computer related to some problem or task concerning visualisation could easily be implemented in the form of one or two obligate homework tasks.

- Sociomathematical norms for changes between different media including computer use. In addition, specifically mathematical beliefs and values particularly concerning the use of computer and the choice between media. This can be seen as an issue within the area of the media essentials.

- The role of visualisation in the ViT project was internal mathematical, as a means for basic, geometrical interpretation of the linear transformations; visualisation was not for example introduced as a means for arguing, or a tool for problem solving (which might have involved the tool essentials). Neither was visualisation seen as shortcut to solution of other tasks or exercises during the course. In this sense, visualisation was seen as mathematical activity suitable to support the students’ formation of certain mathematical conceptions. This points to issues which are dealt with in the frame of the vehicle for learning essentials.

- Since linear algebra is commonly acknowledged to be a cognitively and conceptually difficult area (Dorier and Sierpinska 2001) the visualisation was linked with the touchiest spot, so to say, of the course.

- It seems reasonable to have let the students have the opportunity to avoid deeper engagement if they wanted to, for example those who studied other subjects than mathematics. On the other hand, it was clear from our study and from the professional mathematicians that the potentials of using visualisations were huge which might suggest that all students should be obliged to go through a tutorial and prove at a certain level that they had generated one or two sites as an instrument in accordance with the theory of instrumental genesis.

Whether the use of ViT sites should be optional or not, the assistant teachers should be in charge of developing the students’ corresponding competencies. Our study points to the need for pedagogical training of the assistant teachers. The major potentials for successful implementation of a project like ViT could be realised during courses for the assistant teachers which took into account i) The four essentials and ii) The process of instrumental genesis. In particular, these courses would help the assistant teachers’ development of pedagogical competence needed to facilitate the students’ learning processes.
II. Teaching materials and ideas

1. Competence aims

The main subject area deals with the fundamental language of symbols in mathematics. Calculation, manipulation and argumentation using mathematical symbols are therefore absolutely central to the main subject area. Argumentation involves the use of different types of proof and logical relations. In addition, the main subject area covers key concepts such as polynomials, polynomial division and rational, logarithmic and exponential expressions.

The main subject area also deals with the analysis and calculation of numerical patterns, finite sums and infinite series. Basic methodologies in the main subject area are recursion and induction. It also focuses on series, convergence and proof by induction.

1.1. Algebra

- work with powers, formulae, brackets and rational and quadratic expressions with numerals and letters
- solve equations, inequality and systems of equations of the first and second degree, in longhand and by digital means (S1 = lower level)
- derive the basic arithmetical rules for logarithms, and use these and the power rules to simplify expressions and solve equations and inequalities (R1 = higher level)
- sum finite series with or without digital means, derive and use the formulae to the sum of the first n members in arithmetic and geometric series, and use this to solve practical problems (R2)
- find patterns in numerical series and use them to sum finite arithmetical and geometrical series and other series, with and without digital means
- determine whether an infinite geometric series is convergent and calculate the sum of the series
- use series to solve practical problems related to savings, loans and hire-purchase (S2)

1.2. Functions

- draw graphs of polynomial functions, exponential functions, power functions and rational functions with linear numerators and denominators with and without digital means
- determine zero points and intersection points between graphs, with and without digital means
- give an account of the definition of the derivative, work out the derivative for polynomial functions and use this to discuss polynomial functions (S1)
- use formulae for the derivative of power, exponential and logarithmic functions, and differentiate composites, differences, products, quotients and combinations of these functions
- draw graphs to functions with and without digital means, and interpret the basic characteristics of a function using the graph (R1)
- derive polynomial functions, power functions, exponential functions and logarithmic functions, and sums, differences, products and quotients of these functions, and use the chain rule to derive combined functions (S2)
- simplify and solve linear and quadratic equations in trigonometric expressions by using relations between the trigonometric functions (R2)

1.3. Modelling

- use first derivative and second derivative to elaborate on and discuss the path of functions and interpret the derivatives in models of practical situations (R1)
• create and interpret functions as models and describe practical problems in economics and social science, analyze empirical functions and use regression to find a polynomial approximation of a function, power function or exponential function (S1)
• solve economic optimization problems in connection with income, cost and demand functions, and calculate and use marginal costs and income in simple models
• model exponential and logistical growth rate by using exponential functions and logarithmic functions (S2)
• formulate a mathematical model with the help of central functions on the basis of observed data, process the model and elaborate on and discuss the result and method
• calculate integrals of the central functions by anti-derivation, substitution, partial fraction decomposition with linear denominators and integration by parts
• formulate a mathematical model with the help of central functions on the basis of observed data, process the model and elaborate on and discuss the result and method (R2)
• transform trigonometric expressions of the type $a \sin kx + b \cos kx$, and use these to model periodic phenomena (R2)

In the following paragraph, we present ideas, tasks and materials for teaching which aim to support the students’ development of the competencies in algebra, functions and modelling.
2. Mini project about Series

2.1. Series, guidelines

Competence aims:
Mat S2 - Algebra.
- find patterns in numerical series and use them to sum finite arithmetical and geometrical series and other series, with and without digital mean.
Mat R2 - Algebra.
- find and analyze recursive and explicit formulae for numerical patterns with or without digital means, and implement and present simple proofs linked to these formulae.
- sum finite series with or without digital means, derive and use the formulae to the sum of the first n members in arithmetic and geometric series, and use this to solve practical problems

Learning Objectives
- Finding the formula for series
- Displaying a mathematical formula using drawings
- Working together in groups, and have the responsibility to explain a mindset for their peers
- Working independently with a type of task the students had never seen before. Discuss and formulate mathematics

Duration and timing
Using the scheme takes 4 hours. Suitable for a whole day.

Method of Working
Students work in groups in two stages. This grouping can also be thought of as a matrix, because students are divided first in one direction, followed by the other. Let’s say that we have 20 pupils and share them 5 groups with 4 people in each group, group 1, ..., 5. Each of the 5 groups dealing with its own unique mission. The group must ensure that all team members understand the task and solution, and is able to explain the solution for others. Secondly, the class is divided in a different way so as to form new groups of members of each of the first 5 groups. There is thus formed four new groups, group A, ..., D, consisting of one person from each of groups 1, ..., 5. In the new group sits 5 people who have worked with 5 different tasks, and which now has the responsibility to explain the solution for the others in the group. To the first part, it is advantageous to assemble groups so students have the same level. As far as possible the same number in each group.

Prerequisites
Requires knowledge of the joints, included sequences series, the formula for nth paragraph, and the formula for the next paragraph. This can be done in two double hours by letting students work with the first two paragraphs (sequences, series) from Aschehoug Mathematics S2, chapter about series. Other materials or textbooks can of course be used, but the advantage of the above material is that it works more investigative out of different types of sequences and series. Once one has first
introduced arithmetic and geometric series, students have a tendency to only look for these patterns and consequently become more locked in their quest. Alternatively, English-language material used, for example, from IB Math Studies.

**Procedure**

- Go through a common Introduction (Completing the Square, Algebraic Areas I). Students work with task 1 (Sums of Integers I, Sums of Odd Integers I). Communal collecting on the board. Duration about one hour.
- Division of students into Matrix groups. Working with Task 2 (Sums of Integers II, Sums of Odd Integers II, Squares and Sums of Integers, Sums of Cubes IV, Geometric Series III). If some groups finished early, they can get extra tasks. Duration about one hour.
- Restructuring of the groups across. Students get all tasks. Students need to explain to each other how their group solved the task. Duration about one hour.
- Common review on the board by students. Duration about one hour.

**Challenge to students**

Extra tasks. Differentiation of tasks to groups.

The last task (Geometric Series III) is difficult. Experience shows that even good students may not get it all, but only parts of the task. They work hard with tasks and welcome the challenge. The teacher goes through the solution with the students.

**Common summary and discussion**

During the work of the last groups, it may be a good idea to talk with students, and ask if they are willing to undergo the task on the board in front of the class. The advantage is that the learner practicing in presenting mathematics in an understandable way. They are a little more confident doing it in front of the class when they have already trained in the group. The common review ensures that any errors, inaccuracies and different perceptions appears, and can provide the basis for further discussions.

**2.2. Miniproject about Series, the tasks**

*Task: Explain how the figures can be interpreted, and use this to see a relationship between figures and expressions:*
A. Completing the Square

\[ x^2 + ax = \left( x + \frac{a}{2} \right)^2 - \left( \frac{a}{2} \right)^2 \]

B. Algebraic areas I

\[ (a + b)^2 + (a - b)^2 = 2(a^2 + b^2) \]
C. Sums of integers I

\[ 1 + 2 + \cdots + n = \frac{1}{2} n (n + 1) \]

D. Sums of Odd Integers I

\[ 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \]
E. Sums of Integers II

\[ 1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2} \]

F. Sums of Odd Integers II

\[ 1 + 3 + \cdots + (2n - 1) = \frac{1}{4} (2n)^2 = n^2 \]
G. Squares and Sums of Integers

\[ 1 + 2 + \cdots + (n - 1) + n + (n - 1) + \cdots + 2 + 1 = n^2 \]
**H. Sum of Cubes**

\[ 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4} [n(n + 1)]^2 \]
I. Geometric Series

\[
(1 - r)^2 + r^2(1 - r)^2 + r^4(1 - r)^2 + \cdots = \frac{(1 - r)^2}{(1 - r)^2 + 2r(1 - r)} = \frac{1 - r}{1 + r}
\]

\[
1 + r^2 + r^4 + \cdots = \frac{1}{1 - r^2}
\]

\[
a + ar + ar^2 + \cdots = \frac{a}{1 - r}
\]
3. Algebra tasks

Competence aims:

- work with powers, formulae, brackets and rational and quadratic expressions with number and letters
- solve equations, inequality and systems of equations of the first and second degree, in longhand and by digital means

Tasks

A. Small tasks

1. Expand these squares to complete the statements:
   a) \((x+6)^2 = \)  
   b) \((x-3)^2 = \)  
   c) \((x-10)^2 = \)  
   d) \((x+2\frac{1}{2})^2 = \)  
   e) \((x+\sqrt{7})^2 = \)  
   f) \((x+a)^2 = \)  
   g) \((x-a)^2 = \)  

2. Describe a pattern in the way that your answers to problem 1 relate to the expressions you were told to expand?

3. Use the patterns to figure out what value \(a\) and \(b\) must have to make the following statements true:
   a) \((x+a)^2 = x^2 + 6x + b\)  
   b) \((x+a)^2 = x^2 - 6x + b\)  
   c) \((x+a)^2 = x^2 + 9x + b\)  
   d) \((x+a)^2 = x^2 + bx + 9\)  
   e) \((x+a)^2 = x^2 + bx + b\)  
   f) \((x+a)^2 = x^2 - bx + 81\)


B. Discussion about discount and VAT

Task: You are in a store and find a product that is advertised with a 20 % discount. The price of the item is listed without VAT. You decide to take advantage of this offer and take the goods with you to checkout. The cashier gives you 20 % off the price, and then adds 14% VAT. You start arguing and say that adding 14% VAT before you subtract 20% will give you a larger discount, and you will pay less for the item. The store manager is called for but he disagrees with you. She says that if you subtract 20% first, the 14% VAT will be calculated from a lower amount and this will give you a better price.

Discuss which method is better for the customer. Add VAT and then deducting the discount, or withdraw from the discount before adding VAT?
Difficulty of the task can be increased by not quantify discount and VAT:

Task: You are in a store and find a product that is advertised with a discount given in percent. The price of the item is listed without VAT. You decide to take advantage of this offer and take the goods with you to checkout. The person at the checkout subtracts the discount and then adds VAT. You protest and say that if you add the VAT before you subtract the discount it will give you a bigger discount and you will pay less for the item. The store manager is called for but disagree with you. She says that if you subtract the rebate first you will pay VAT of a lower amount and this will give you a better price.

Discuss which method is better for the customer. Add VAT and then deducting the discount, or withdraw from the discount before adding VAT?

(The idea is from Movshovitz-Hadar, Nitsa m.fl. One Equals Zero,)

C. Understanding an algebraic expression: The Free Throw problem

Task: Megan is a basketball player and has made 25 of 40 free throws for her team this season. There are three more games left. If Megan shoot 20 free throws in the remaining three games, which of the following equations determines x such that X is the number of free throws she must score to having succeed in 70% of their free throws?
Explain to each other how you made your decision, and what makes the other alternatives wrong.

\( a) \)
\[
\begin{align*}
20 + x &= 7 \\
40 + 25 &= 10
\end{align*}
\]

\( b) \)
\[
\begin{align*}
25 + x &= 7 \\
40 + 20 &= 10
\end{align*}
\]

\( c) \)
\[
\begin{align*}
40 + 20 &= 7 \\
25 + x &= 10
\end{align*}
\]

\( d) \)
\[
\begin{align*}
25 + 20 &= 7 \\
40 + x &= 10
\end{align*}
\]

\( e) \)
\[
\begin{align*}
25 + x &= 7 \\
40 + 20 &= 10
\end{align*}
\]

\((NCTM\text{-}journal,\text{Mathematics Teacher})\)
4. Tasks about functions

A. Using GeoGebra in the introduction of the derivative

Introduction

Norwegian textbooks in mathematics usually introduce the derivative using a geometric approach. When students have become familiar with this, introduces the algebraic approach. Finally, students are introduced to the rules for differentiating polynomials.

The course start with average rate of change, introduces so instantaneous rate of change, where instantaneous rate of change when \( x = a \) is the slope of the tangent \( \frac{\Delta y}{\Delta x} \).

Finally, the derivative is derived using the definition. Here, we start by looking at how the subject is introduced in today's textbooks:
In Norwegian high school, all students bring their own computers. It is therefore natural to use GeoGebra to introduce the derivative. In the following we will show a way to introduce differentiation of $f(x) = x^2$ and $f(x) = x^3$.

First students will explore the relationship between the value of $x$, and the slope of the tangent and find its derivative. Furthermore, students should find the connection between a given $x$-value and the derivative of the function for this value of $x$.

**Exploring the curve** $f(x) = x^2$
The derivative

Draw the tangents to the curve in the given points and find the value of the slope to each of the tangents.

Make a table with each value of $x$ and the value of the slope to each of the corresponding tangents.

Try to find an expression of the derivative $f'(x)$ using these values.

The derivative

Draw the tangents to the curve in the given points and find the value of the slope to each of the tangents.

Make a table with each value of $x$ and the value of the slope to each of the corresponding tangents.

Try to find an expression of the derivative $f'(x)$ using these values.
In this activity students worked together in pairs. They should fill in the table and try to find a pattern so that they could find an expression for the derivative of the function \( f(x) = x^2 \).

**x-value**

**Value of the slope**

The students worked on this task in about 20 minutes. Then they had to write down the expression they had found for the derivative of the function \( f \). All the proposals were listed on the board, and most had found the right expression \( f'(x) = 2x \).
**Exploring the curve** \( f(x) = x^3 \)

**The derivative of \( x^3 \)**

Enter points from \( x = -3 \) to \( x = 3 \) and the corresponding value of the derivative. Do it the following way: (3, f(3)), etc.

Can you identify the curve given by these points?

![Graph of the function \( f(x) = x^3 \)](image_url)

**The derivative of \( x^3 \)**

Enter points from \( x = -3 \) to \( x = 3 \) and the corresponding value of the derivative. Do it the following way: (3, f(3)), etc.

Can you identify the curve given by these points?

![Another graph showing points on the curve](image_url)
In this activity students worked together in pairs. They should fill in the table and try to find a pattern so that they could find an expression for the derivative of the function \( f(x) = x^3 \)

<table>
<thead>
<tr>
<th>x-value</th>
<th>Value of ( f'(x) )</th>
</tr>
</thead>
</table>

The students worked on this task in about 20 minutes. Then they had to write down the expression they had found for the derivative of the function \( f \). All the proposals were listed on the board, and only a few had found the right expression \( f'(x) = 3x^2 \). Students found three different expressions for the derivative, \( f(x) = x^3 : f'(x) = 3x \), \( f'(x) = x^2 \) or \( f'(x) = 3x^2 \). After a class discussion, we agreed that the correct expression was \( f'(x) = 3x^2 \).

We continued to differentiate \( f(x) = x^4, f(x) = x^5, f(x) = x^6 \) .... We could conclude that the derivative of \( (x')' = rx^{-1} \) was correct.

**Summing up:**

- The students worked hard trying to find a pattern from the two tables and a pattern that described the expression for the derivative
- Most of them learned this rule of differentiation: \( (x')' = rx^{-1} \)
B. Experiences from using CAS to integrate functions by partial integration, change of variables and partial fractions

Introduction

The students were given a sheet with different function expressions. They should use CAS (www.wolframAlpha.com or GeoGebra) to find integrals of function expressions. They were asked to look for patterns in the answers, and find out how and why the integration of the different expressions gave different answers. In the beginning some of the students were unwilling to integrate expressions using CAS. They believed that to use digital tools in this way was a form of cheating, and that the answers did not make sense. Gradually students began to see some patterns when the method with variable shift was used. In class discussion that followed, revealed several proposed solution method on the integral of \( \int \frac{2x}{x^2 - 4} \, dx \), and that we could put \( u = (x^2 - 4) \) and thus transform expression \( \int \frac{2x}{u} \, dx = \int \frac{u'}{u} \, dx \). By deriving \( u \) and introduce \( u' = \frac{du}{dx} \) we noticed the connection \( u' = \frac{du}{dx} = 2x \Rightarrow du = 2xdx \), thus we could write the following:

\[
\int \frac{2x}{u} \, dx = \int \frac{u'}{u} \, dx = \int \frac{1}{u} \, du .
\]

We had found the pattern to the method with variable shift. It was difficult to find patterns of the other two methods, they may be derived by calculation.

When the students had worked with integration methods in about 1 week, they had a test in integrating expression where they should use all three methods. The result of the test gave an impression that most students were capable to integrate when using the method with variable shift. There were only a few students who also managed the 2 other methods, most mixed method with partial integration and method with variable shift. Naturally, it is not possible to draw any conclusions from this little test. After have worked more with integration, taught most students to use all integration methods. I believe that adopting digital tools also in the learning phase, can be
useful. When students find the answer without account first, and must try to find a pattern itself, they may be more active in the learning phase. My experience of this little project is that when students began to work on finding patterns and found a pattern, they were also interested to find out whether they had found the right expression.

*Introducing integrals*

Task: Use www.wolframAlpha.com or GeoGebra and find the integrals:

<table>
<thead>
<tr>
<th>Nr</th>
<th>Integre</th>
<th>Svar</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\int \frac{1}{x-2} , dx$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\int \frac{2x}{x^2-4} , dx$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\int xe^{-x^2} , dx$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\int xe^{-x^2} , dx$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$\int x^2 e^x , dx$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$\int \frac{\ln x}{x} , dx$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$\int \sqrt{2x+1} , dx$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$\int x \cdot e^{-x} , dx$</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>$\int x^2 \cdot \ln x , dx$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$\int 2x \ln x^2 , dx$</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>$\int 2xe^x , dx$</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$\int 2xe^{x^2+1} , dx$</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>Finn arealet mellom grafen til funksjonen $f(x) = \frac{1}{x} - x$ og x-aksen når x ligger mellom 0,5 og 1</td>
<td></td>
</tr>
</tbody>
</table>
Task: Try to look for patterns in the answers and identify which type of exercises that provides different types of answers.
C. How to turn an ordinary problem into a more open and exploring task

Below you will find a list of different quadratic equations listed with four alternative solutions. Traditionally you are asked to mark the right solution. Instead of doing this we can ask the students to pick one or two of the incorrect answers and challenge them to reorganise the equation so that an incorrect answerer will be the correct one.
4. Modelling

A. Understanding a graph

Task: A beautiful winter’s day I decide to go cross country skiing. I’m a little curious about how much time I will use, so I decide to take notes along the way.

The first 30 minutes I ski 4.5 km. I ski for 2.5 hours more before I decide to take a break. While enjoying the sun and the lunch I brought, I realise I have skied another 12 km. After 25 minutes I start the return. I choose another trail back to the lodge and after 30 minutes I have skied 3.5 km. Here I see a signpost telling me I have 15 km left before I’m home. On these last 15 km it turns out that I use 35 min.

Draw a graph to illustrate my trip.

Questions to be discussed:

How long does the trip?
How many km I've gone totally?
Can you imagine / suggestions about the terrain, I have gone in?
How far I've gone halfway through?
Are there any other interesting aspects to discuss or tell from the graph and information above?

Figure i: The teacher's suggestion

Figure j: The teacher's suggestion
Experiences
The students worked with the graph for about 20 Min. They had to write down their thoughts and discussions. Here are some of their suggestions:

“We were not sure whether we would let the graph go back down to zero or if it should continue to rise. We decided to let it fall. We tried this, but when our graph was down to 8.5 km and she met a sign that told her she had 15 km left we had to change our opinion and started to let it rise. We understood that she did not “lose” km.”

An example where the students have used x-axes as distance, and y-axes as time. They might also have had difficulties with implementing the break into the graph.
This group have made a correct plot, but instead of drawing the graph from point to point they have drawn the straight line that fits best:

![Figure m](image)

“The graph is steeper when she goes downhill, that means the last 15 km where she has a higher average speed.....after the break there is most likely flat or slightly upwardly.....

....our graph illustrates that the steeper terrain the gentler is the slope of the graph. This is because she uses more time on a shorter distance. Where the graph is flat she has a break.”

![Figure n](image)
B. Biking

Task: Peter’s bike ride lasts 30 minutes. Describe his trip:

![Graph of Peter's bike ride](image)

C. The slope of a linear function

Task: Imagine that you have a ladder with a fixed length that’s leaning against a wall. Suppose you move the ladder so that it now ranks exactly twice as high up on the wall as it did in the first place.

Make a drawing of the situation.

The slope of the ladder is then...

A) Less than twice what it was
B) Exactly twice what it was.
C) More than twice what it was.
D) The same as it was before.
E) There is not enough information for us to determine whether any of a) - d) are correct.

(NCTM-journal, Mathematics Teacher)
D. Nordic World Ski Championships in Falun

In this task we will study how the temperature has changed in the Northern Hemisphere from 1881-2014, and see how temperatures have fluctuated over the years it has been arranged Nordic World Championship in Falun, Sweden.

Tasks:

1. On the next page you find a list of temperature data. Use GeoGebra or another dynamic geometry software to make a graph that fits these data. Y values providing temperature deviation (from the average of the period 1951 to 1980) for the winter months (Jan-feb) in the northern hemisphere. X values provide years from 1881. Describe the temperature development in broad terms.

2. Use a dynamic geometry software to find the straight line that fits the data points. What is the equation of this line?

3. When is the temperature deviation 2 °C, according to this line?

4. Ski Championships have been held in Falun four times: in 1954, 1974, 1993 and 2015. For the winter months these years, the average temperature deviation was respectively 0, -0.17, 0.17 and 0.96 °C. Add Point and name (year) to these points. Which WC has been the warmest and which has been the coldest? What is the temperature difference between these?
5. Now use only the data from the four years it has been arranged World Championship in Falun, and create a new regression. Compare the lines. What does the slope tell you, and why are they different?

6. We notice that there has been arranged Championships in Falun approximately every 20 years since 1954. We assume that there will be arranged a new WC in Falun in 2035. What will the temperature deviation be by 2035, according to your models? How realistic is this? Discuss.

(https://skolelab.uib.no)

<table>
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<th>Year</th>
<th>Temperature deviation</th>
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<tbody>
<tr>
<td>1881</td>
<td>-0.29</td>
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<tr>
<td>1882</td>
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<td>1883</td>
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E. What will happen with the 10,000-meter run?

Task: The 10,000 metres or 10,000-meter run is a common long-distance track running event. The event is part of the athletics programme at the Olympic Games and the World Championships in Athletics and is common at championship level events. It is less commonly held at track and field meetings, due to its duration. The 10,000 metres track race is usually distinguished from its road running counterpart, the 10K-run, by the referring to the distance in metres, rather than kilometres.

The 10,000 metres is the longest standard track event. The international distance is equal to approximately 6.2137 miles (or, approximately 32,808.4 feet). Added to the Olympic program in 1912, athletes from Finland, nicknamed the “Flying Finns”, dominated the event until the late 1940s. In the 1960s, African runners began to come to the fore. In 1988, the women’s competition debuted in the Olympic Games.
Below we have listed times for the season’s best men and women from 1970 – 2015.

Do some data analysis on the findings. Discuss with your partner what you think is the best ways of interpreting and showing the results. What do you think about future developments and will women ever run faster than men at 10 000m?

(pictures and charts from https://en.wikipedia.org/wiki/10,000_metres)

### Men

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(pictures and charts from https://en.wikipedia.org/wiki/10,000_metres)
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Using sliders to investigate functions, tangents and integrals
Using sliders to investigate functions, tangents and integrals

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Introduction

This chapter is on the use of the software GeoGebra when investigating the shape of graphs of functions. There are many possibilities to do this; the derivative of a function can be calculated directly with a command that also gives its graph, it can be constructed using a tangent tool and finally there are possibilities of altering parameters (by using sliders) in the definition of a function and watch its derivatives change dynamically. Similar techniques can be used to study the integral of a function in particular in applications.

There are several examples given as well as some tasks for students to solve.

Derivatives and tangent lines

In many textbooks on differentiation there are exercises of the type where a student is asked to find the equation of the tangent line to the graph of a given function at a given value, e.g. at $x = 1$. Usually these problems are solved by differentiating the function, finding the value of the derivative at the given $x$ – value e.g. $f'(1)$, to get the slope of the tangent line, and then finding the equation of a line through the given point e.g. $(1, f(1))$. So the requested answer is the equation:

$$y = f'(1)(x - 1) + f(1)$$

Software like GeoGebra or a graphics calculator is very useful in such situations to check if the calculated answer makes sense and also to connect the algebraic calculations just completed to the actual geometric object that has been found. It is possible to solve the problem in GeoGebra in two different ways, by computing the derivative and its value or by using the tangent tool. The derivative can be computed by writing `Derivative[f]` or $f'(x)$ in the input field. Its value, at a given $x$ – value is obtained by writing e.g. $f'(1)$ in the input field. It is also possible to use the tangent tool to get the tangent at a certain point, directly. If that is done it is necessary to first define the point on the graph of the function by writing e.g. $(1, f(1))$ in the input field or by using the point tool and clicking on the graph of the function.
Constructing the graph of the derivative point by point

The graph of the derivative of a function \( f(x) \) can also be constructed point wise by using the tangent tool. For each value of \( x \) the corresponding point on this graph is \( (x, f'(x)) \). If we have the graph of the function \( f(x) \), mark a point \( A \) on the graph and use the tangent tool to get the tangent, \( a \), at that point, we can use the command \( \text{Slope}[a] \) (or a corresponding tool) in GeoGebra to get the value of the slope. This gives one point on the graph of \( f'(x) \), namely \( B = (x(A), \text{Slope}[a] = f'(x(A))) \) where \( x(A) \) is the \( x \)-coordinate of the point \( A \). If we now use the mouse to drag the point \( A \) along the graph of \( f(x) \) we get corresponding points \( (x(A), f'(x(A))) \) on the graph of the derivative.
The value of the derivative at a point is the slope of the tangent line.

When we follow the procedure above, we only get one point, on the graph of the derivative, at a time. If we right click on the point B and select Trace on and then move the point A along the graph of $f(x)$ the point B will move accordingly and leave a trace that is the graph of the derivative on an interval.

**Fig. 2** The value of the derivative at a point is the slope of the tangent line

**Fig. 3** The trace of the point $B = (x(A), Slope[a] = f'(x(A))$)
GeoGebra can give extremal points of polynomials and can compute numerically extremal points for other functions if an interval is given (the command is Extremum) and there is also a command that gives points of inflection for polynomials (InflectionPoint).

**Finding a third degree polynomial with a certain slope**

A different kind of problem is of the type:

The slope of the tangent to \( f(x) = ax^3 - 3x^2 + x + 3 \) at the point \((-1, f(-1))\) is 1.

Find the value of \( a \).

These problems are usually solved algebraically only, i.e. the student differentiates the function, substitutes the value \(-1\) for \( x \) and solves \( f'(−1) = 1 \) to get the value of \( a \):

\[
f'(x) = 3ax^2 - 6x + 1 \quad \text{which gives} \quad f'(−1) = 3a + 6 + 1 = 1 \quad \text{so} \quad 3a = −6 \iff a = −2.
\]

Once the problem has been translated to an algebra problem it is often not considered necessary to graph the function or check the answer.

There are also in most calculus books more complicated problems of the same type involving the location of relative minimum and maximum, points of inflection, integrals etc. Most often the assumption is that the calculations are done by hand and the problem is converted to an algebra problem, i.e. to solving a system of equations. The focus is then more on algebra than on the actual shape of the function in question.

Below we shall consider ways to use GeoGebra to solve such problems in a more graphical way. Many of the tasks we consider come from material used at a gymnasium in Iceland (Stærðfræði 403, Menntaskólinn við Hamrahlíð, 2010).

**Using sliders**

To solve the problem above graphically we might try to graph the function \( f(x) = ax^3 - 3x^2 + x + 3 \) for several different values of \( a \), graph the tangents at the point \((-1, f(-1))\) to try to find the one with the desired slope.
This would be quite time consuming when solving more complicated problems.

Using GeoGebra it is very easy to investigate the effect of changing the value of one (or more) parameters occurring in the definition of a function such as the one above. To define such a parameter select the slider tool and click on the Graphics view. When that is done a small window opens:

![Slider Window]

To solve the example given above we create a slider $a$ and then define the function above by writing $f(x) = ax^3 - 3x^2 + x + 3$ in the input field.
Fig. 6 The function $f(x)$ with the original value of $a = 1$. The slope of the tangent line at $(-1, f(-1))$ is 10.

In the figure above we have first defined the slider $a$, then the function $f(x)$. We then define the point $(-1, f(-1))$ in the input field and use the tangent tool to get the tangent line to $f(x)$ at that point. The slope tool is used here to mark the slope of the tangent line. To find the desired value of $a$ we simply move the slider until the value of the slope is 1.

Fig. 7 The value $a = -2$ gives the desired slope
Task: The graph of the function \( f(x) = x^2 + bx + c \) has a minimum at the point \((-3,-10)\) and goes through the point \((0,-1)\). Find the values of \( b \) and \( c \) by using sliders in GeoGebra (you will need to change the interval that \( b \) is defined on, this is done by right clicking on the slider and selecting Object Properties).

Task: We have given the function \( f(x) = x^2 - bx + c \). The tangent to \( f(x) \) at the point \((-2,0)\) is perpendicular to the line \( y = x - 3 \). Find the values of \( b \) and \( c \) using sliders in GeoGebra. Note that you have to think about the order in which you change the value of the sliders (see fig. 8).

![Graph of \( f(x) = x^2 - x + 1 \) with a tangent line through the point \((-2, f(-2)) = (-2,7)\). The line \( y = x - 3 \) together with a perpendicular line that goes through the point \((-2,0)\).](image)

**Fig. 8** Graph of \( f(x) = x^2 - x + 1 \) with a tangent line through the point \((-2, f(-2)) = (-2,7)\). The line \( y = x - 3 \) together with a perpendicular line that goes through the point \((-2,0)\).

**Third degree polynomial**

The problem below is from a handout for students at the gymnasium level [3]. It is assumed that the students will solve it using algebraic techniques after differentiation.

The objective is to find a third degree polynomial \( f(x) = x^3 + bx^2 + cx + d \) that satisfies the following conditions:

The function has a minimum at \( x = 1 \) and its graph has an inflection point at \((-1, 12)\).

Below we take you through a number of steps to solve this problem using sliders in GeoGebra.

1. Define three sliders \( b \), \( c \) and \( d \). Define the third degree polynomial \( f(x) = x^3 + bx^2 + cx + d \) in the input field.
We can make this problem a little easier by making sure that the sliders only have integer values. To do this, right click on the slider and choose Object Properties and under the tab Slider set the Increment equal to 1.

2. Experiment with the sliders and try to find values such that the function satisfies the given conditions. In particular, what is the effect of changing the value of \( d \)?

3. Write \( f'(x) \) in the input field, right click on the graph shown and choose Properties and change the color of the graph of \( f'(x) \) so that it is easier to distinguish it from the graph of \( f(x) \). Write \( f''(x) \) in the input field and choose a third color for the appearing graph.

4. Change the value of \( c \) and observe its effect on the graph of \( f'(x) \).

5. Change the value of \( b \) and observe its effect on the graph of \( f''(x) \).

6. We have given that \( f \) has a point of inflection at \( x = -1 \). What does this tell us about the graph of \( f''(x) \)? Can you find a value of the slider \( b \) to satisfy this condition?

7. We have given that \( f(x) \) has a minimum at \( x = 1 \). What does this tell us about the graph of \( f'(x) \)? Can you find a value of the slider \( c \) so that this is satisfied? Note that you might need to change the settings of the slider \( c \), this is done by right clicking on \( c \) and choosing properties.

Now the graph of \( f \) should have the desired shape but we still need to find a value of \( d \) that gives the correct location of the graph.

8. Define the point \((-1, 12)\) in the input field. Change the value of \( d \) until the graph of \( f(x) \) passes through the point. If this is difficult to see graphically you could write \( f(-1) \) in the input field to get an exact value (in the algebra window) and then move \( d \) until that value is 12.

9. Now we should have found the correct values of \( b, c \) and \( d \) and the equation of the function \( f(x) \) should be in the algebra window.
Task: If we are now asked to change the function such that the point of inflection is at $(-1, 10)$ which slider do we need to work on to accomplish that?

Task: What if the new information is that the point of inflection is at $(-2, 10)$?

Task: Can you change the values of the sliders such that $f(x)$ does not have any extremal points? How?

Task: Can you change the values of the sliders such that $f(x)$ does not have any inflection point? How?

A traditional way of solving the original problem would be to differentiate $f(x)$ twice and then use the information on the inflection point to get the equation $-6 + 2b = 0$, the information on the minimum to get $3 + 2b + c = 0$ and finally we get $-1 + b - c + d = 12$ since $(-1, 12)$ is on the graph.

Task: Define one more slider $a$ and redefine the polynomial as $f(x) = ax^3 + bx^2 + cx + d$. Set up the system of equations (with pencil and paper) we get if this function is to satisfy the conditions above. You should get 3 equations in 4 unknowns so there is a family of functions that satisfies the conditions. Can you redefine the parameters $b, c$ and $d$ in terms of $a$ to get all third degree polynomials that satisfy the conditions?

Task: Make up a similar problem for a fourth degree polynomial.
Task: Make up similar problems using other types of functions e.g. exponential and logarithmic functions, trigonometric functions etc.

Problems with this solution method

Not every problem of this kind is this easy to solve. If we try to solve the similar problem:

Find the value of $a$ such that $f(x) = ax^3 - 2x^2 - x + 5$ satisfies $f'(3) = 3$,

in the same way we do not get the solution as easily as in the previous examples.

Once we start doing what we did before we notice that it is not so easy to get a slope with exactly the value 3 i.e. the slope jumps between big and small values. It is therefore easier to click on the point on the slider and use the arrow keys on your keyboard to change the value of the slider. Doing this you see that the slope jumps between values a little bigger and a little smaller than 3. It is therefore wise to tune the slider more precisely. To do that we right click on $a$ and choose Object properties and set the increment to 0.0001. Now we of course need to make sure that GeoGebra is actually using enough decimals; this is done under Options and then Rounding. With this increment we get that $a = 0.5926$ gives the slope 3.0002 and $a = 0.5925$ gives the slope 2.9975. The correct value of $a$ is therefore somewhere between those two.

Fig. 10 Approximate solution

Task: Try to explain what is happening. Hint: solve the problem algebraically.

Task: Make the increment even smaller and try to solve the problem.

If we have enough patience to work on this using more decimals and a smaller increment we should eventually get that $a = 0.592592$ seems to get us very close to the desired values of the slope. The number appears close to a periodic decimal which we can convert to the
fraction \( a = \frac{592}{999} = \frac{16}{27} \) which is exactly the solution we get from solving the problem algebraically.

**Newton’s law of cooling**

Newton’s law of cooling states that the rate of change of temperature of an object is proportional to the difference in temperature between the object and the surrounding medium. This leads to the differential equation \( \frac{dT}{dt} = -k(T - T_0) \) where \( T \) is the temperature of the object, \( T_0 \) is the temperature of the surrounding medium and \( k \) is a constant. This equation has the solution

\[
T = ce^{-kt} + T_0,
\]

where \( c \) is a constant.

So if a cup of coffee at the temperature 95°C is placed in a room at 20°C (= \( T_0 \)) and 5 minutes later the coffee has the temperature 85°C then we can use that information to find the values of \( c \) and \( k \) and thus get a model that will help us estimate when the coffee will be drinkable (say at 75°C).

This kind of a problem is usually solved algebraically, that is, we put the given information into the formula to get equations to solve:

when \( t = 0 \) we get \( 95 = ce^0 + 20 \) so \( c = 75 \) and the formula is \( T = 75e^{-kt} + 20 \).

when \( t = 5 \) we get \( 85 = 75e^{-k5} + 20 \) which gives us \( k = \frac{-1}{5} \ln \left( \frac{65}{75} \right) \approx 0.0286 \)

We therefore have \( T = 75e^{-0.0286t} + 20 \) and we can use this to find out when the coffee will have the temperature 75°C (in approximately 11 minutes).

We can easily make a picture of this in GeoGebra:
Instead of the calculation of \( k \) above we could have created a slider for \( k \) and then changed its value until the curve would pass through \((5, 85)\).

We can make this problem more interesting by varying the input information using sliders for \( T_0 \), the temperature for surrounding medium, \( T_c \) for the temperature of the object at the beginning, \( T_m \) for the information on temperature after \( m \) minutes and finally a slider \( m \) for minutes. A formula for \( k \) in terms of the given parameters is easy to write down and put in the input field of GeoGebra:

\[
k = \frac{-1}{m} \ln \left( \frac{T_m - T_0}{T_c - T_0} \right)
\]

**Task:** Put the information above into GeoGebra. Assume now that you put your coffee cup on the windowsill where the temperature is \(15^\circ C\) and after only 3 minutes the temperature is down to \(85^\circ C\). When will the coffee be ready to drink?

**Task:** Reuse your model above for cooling of a Coca-Cola can that starts at room temperature. The temperature of a refrigerator is \(2 - 3^\circ C\) and the ideal temperature of Coca-Cola is \(3 - 6^\circ C\) (this is of course a matter of taste). Make some assumption for the values of \( m \) and \( T_m \) (you might want to experiment with this and compare with reality) and find out for how long you need to leave your Coca-Cola in the refrigerator. Now replace the refrigerator with the freezer (\(-18^\circ C\)) and solve the problem for that situation. Put both graphs in the same Graphics view to compare.

**Sliders and Integration**

Simple problems of integration can be solved directly in GeoGebra i.e. we can define a function and use the command \( \text{Integral}[f(x)] \) to get a primitive function and the command \( \text{Integral} [f(x), a, b] \) to get the definite integral from \( a \) to \( b \). If we have two functions \( f(x) \) and \( g(x) \) then \( \text{IntegralBetween}[f(x), g(x), a, b] \) gives the integral of \( f(x) - g(x) \) from \( a \) to \( b \).

Typically sliders are used in this context to demonstrate how the values of the upper-sum and lower-sum approach the value of the integral when the number of intervals increases.
We can also use sliders to define dynamic intervals of integration.

\[ f(x) = x^3 - 19x^2 + 95x - 77 \]

**Fig. 12** Upper- and lower-sums and integral of a function

**Fig. 13** The integral of \( f(x) = x^2 - 3x + 4 \) from 1 to \(-0.5\) is \(-5.25\).
This can be useful to study certain problems of integration, say we have given a function $f(x)$ and we want to answer the question over which interval the integral of $f(x)$ equals a certain number. A typical way of solving such a problem would be to integrate $f(x)$, substitute boundaries $a$ and $b$ and then solve algebraically (there will, of course be many possible answers).

Using sliders $a$ and $b$ we can make GeoGebra compute the integral and then change the values of $a$ and $b$ until we get the desired value.

Below is an exercise from a Calculus book by Greenwell, Ritchey and Lial (2003):

Pollution begins to enter a lake at time $t = 0$ at a rate (in gallons per hour) given by the formula

$$f(t) = 10(1 - e^{-0.5t})$$

where $t$ is the time in hours. At the same time, a pollution filter begins to remove the pollution at a rate

$$g(t) = 0.4t$$

as long as pollution remains in the lake.

![Fig. 14 Pollution](image)

The tasks in the problem are to a) determine the amount of pollution after 12 hours b) use a graphing calculator to find the time when the rate at which the pollution enters the lake equals the rate the pollution is removed c) find the amount of pollution in the lake at the time found in part b. d) Find the time when all the pollution has been removed from the lake.
At $t = 25$ the amount of pollution entering the lake is equal to the amount of pollution being removed

$\int_0^{12} (10(1 - e^{-0.5x}) - 0.4x) dx = 71.25$

Pollution in the lake at $t = 12$ is:

Part a is solved by integrating $f(x) - g(x)$ from 0 to 12 and part b by finding the point of intersection for the two graphs. Part c is solved by integrating from 0 to 25, which gives 105 gallons of pollution. After $t = 25$ hours the pollution is being cleaned up at a faster rate than it enters so the amount of pollution decreases, for instance at $t = 30$ hours there are 100 gallons of pollution. To find out when all the pollution has been cleaned up we need to find the upper limit $L$ of the integral such that

$$\int_0^L (f(t) - g(t)) dt = \int_0^L (10(1 - e^{-0.5t}) - 0.4t) dt = 0$$

This can of course be solved by integrating the function by hand and solving for $L$ but it is very convenient to use a slider here, i.e. we define a slider $L$, calculate the integral from 0 to $L$ and move the slider until we get an answer that is sufficiently close to 0.

Fig. 15 Part a and b of the pollution problem
Fig. 16 After 48 hours the cleanup process has caught up with the pollution and cleaned up the accumulated pollution.

Task: Solve the problem above if the pollution rate is doubled.

Task: Assuming that the cleanup rate is linear, what does it have to be so that we can get the situation under control in less than 24 hours? (hint: define a slider for the coefficient of \( g(x) \)).

Acknowledgment

The first version of the chapter has been included in the project COMENIUS - 510028-LLP-1-2010-1-IT-CO: DynaMat outcomes; available on the project webpage www.dynamathmat.eu.

References


Stærðfræði 403, Menntaskólinn við Hamrahlíð (2010).
Introduction
Using geometric computer programs such as GeoGebra it is very easy to do constructions with a ruler and a compass. In particular, it is easy to draw arcs and circles and putting those together in certain ways one can create objects with different shapes such as eggs and spirals. Much of this work is based on the book Mathographics by Robert Dixon (Dixon 1987).

Arcs and circles
To define a circle we need two parameters, the centre point and the radius. What do we need to define an arc? An arc is basically a part of a circle, so we need the same information to know which circle the arc is a part of. But we also need information on the length and position of the arc on the circle, i.e. given the circle we need to know where the arc starts and where it ends.

![Arc on a circle](image)

**Fig. 1** Arc on a circle

In GeoGebra there are several tools to create circles and arcs:
**Task:** Create several arcs and circles in GeoGebra. Make some of the arcs join each other and use different colours and styles (fillings) to make a pretty picture. To change the colours and styles you right click on an object and select “object properties” in the menu that opens.

Notice that some of the arcs in the picture are joined in a smooth way, that is there is no bend or break where they meet. We now investigate how this is done.

**Task:** Open GeoGebra and create two arcs $c$ and $d$ that meet in one point. Try to move the points defining the arcs in such a way that the meeting is smooth. This task is easier if you draw a line through the meeting point and the center point defining one of the arcs.
The point $D$ can be moved such that the arcs meet in a smooth way resulting in the picture below.

You will probably notice that a necessary condition for smoothness is that the two center points (of the circles defining the arcs) and the meeting point lie on a line. This condition is also sufficient if in your picture the two arcs lie on the opposite sides of the line.
Using this principle we can create pictures like:

![Many arcs meeting in a smooth way](image)

**Fig. 6** Many arcs meeting in a smooth way

This is done with the help of many lines and circles that are hidden in the final picture but shown in the picture below. Because of the dynamic properties of GeoGebra it is possible to move the points around to get different smooth pictures as long as the requirements are satisfied (the three points are on the same line and the arcs are on the opposite sides of that line).

![Many arcs meeting in a smooth way with the help-lines shown](image)

**Fig. 7** Many arcs meeting in a smooth way with the help-lines shown

*Task*: Create your own picture similar to the one above.

**Eggs**

Below you see a picture of several bird eggs. As you can see they are very different in size but similar in shape although some of them are more pointed than the others.
We can use arcs to construct egg-shaped objects like the ones in the picture below. How to do this will be explained in the next section.

**Fig. 9** Moss egg, Four-point egg and Five-point egg created using GeoGebra

**Euclidean Eggs**

In the very nice book *Mathographics* by Robert Dixon there is a section on egg-shaped ovals constructed using a ruler and a compass. The author calls them *Euclidean eggs* and gives pictures of several such eggs although the details of the construction are not given (Dixon (1987), pp. 3 – 11).

Using arcs and circles we can construct these egg shaped curves that resemble cross sections of real eggs. The complexity of these constructions varies greatly. They are relatively easy once the principle has been figured out but the constructions are done in many steps and are quite time consuming.
Moss Egg

Task: Try to use the picture below to construct your own Moss egg:

Fig. 10 Moss egg constructed in GeoGebra

Fig. 11 Screenshot from GeoGebra with the construction protocol

In the construction above the radius of the two large circles is equal to the diameter of the first circle drawn. If we create new points $H$ and $I$ on the diameter line and go through the construction we get an egg that looks slightly different.
Task: Make a GeoGebra worksheet such that you can vary the location of the center points mentioned above and thus experiment with different shapes of a similar type of egg (hint: use a slider).

![Fig.12 Variations of a Moss egg](image)

**Four-point egg**

We are now going to draw the so called *Four-point egg*.

<table>
<thead>
<tr>
<th>Icons</th>
<th>Construction</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Draw a line" /></td>
<td>Draw a line. It is easiest to draw a vertical line e.g. the y – axis although coordinates are not needed for this construction.</td>
<td>a</td>
</tr>
<tr>
<td><img src="image" alt="Define a point" /></td>
<td>Define a point on a</td>
<td>A</td>
</tr>
<tr>
<td><img src="image" alt="Define a line" /></td>
<td>Draw a line through A perpendicular to a</td>
<td>b</td>
</tr>
<tr>
<td><img src="image" alt="Define another point" /></td>
<td>Define another point on a</td>
<td>B</td>
</tr>
<tr>
<td><img src="image" alt="Draw a circle" /></td>
<td>Draw a circle through B with center A</td>
<td>c</td>
</tr>
<tr>
<td><img src="image" alt="Mark an intersection" /></td>
<td>Mark the (other) intersection point of a and c</td>
<td>C</td>
</tr>
<tr>
<td><img src="image" alt="Draw a circle" /></td>
<td>Draw a circle through B with center C</td>
<td>d</td>
</tr>
<tr>
<td><img src="image" alt="Mark intersection points" /></td>
<td>Mark the intersection points of c and b</td>
<td>D, E</td>
</tr>
<tr>
<td><img src="image" alt="Draw a line" /></td>
<td>Draw a line through D and C</td>
<td>e</td>
</tr>
<tr>
<td><img src="image" alt="Draw a line" /></td>
<td>Draw a line through E and C</td>
<td>f</td>
</tr>
</tbody>
</table>
Mark the intersection point of d and e  
Mark the intersection point of d and f  
Draw a circle through F with center D  
Draw a circle through G with center E  
Mark the intersection points of g and the line b  
Mark the intersection points of h and b  
Draw an arc with center C through G and F  
Draw an arc with center D through F and I  

Now your construction should be approximately as shown in the picture below.

![Diagram](image_url)

**Fig.13** The points C and D are two of the four points defining the egg. The green and the red arc are parts of the egg corresponding to the two points.

Four-point egg continued

<table>
<thead>
<tr>
<th>Construction</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Draw a circle with center A through I and J</td>
<td>q</td>
</tr>
<tr>
<td>Mark the intersection point of a and q</td>
<td>L</td>
</tr>
<tr>
<td>Draw a line through I and L</td>
<td>i</td>
</tr>
</tbody>
</table>
Draw a line through J and L

Draw a circle with center J through I

Draw a circle with center I through J

Draw a circle with center L through B

Mark the intersection point of the circle r and the line i

Mark the intersection point of the circle s and the line j

Draw the arc with center J through I and N

Draw the arc with center L through N and P

We now have almost completed the construction of the four-point egg. The four points defining the egg are C, D, J and L as can be seen in the picture below where color has been used to show the correspondence between the arcs and the points.

**Fig. 14** The four points and corresponding arcs.

The remaining arcs of the egg are drawn by symmetry and if we hide all labels, circles and most lines (replacing some of them with segments) we get the final egg as shown below.
Fig. 15 The four point egg

Note: it is not really necessary to draw all the circles used in the construction above since in some cases we can draw the arc directly.
The Five-point egg

Below we have a picture of a five-point egg with all help lines and circles needed to draw the egg.

Fig. 16 Five-point egg

The five orange points are the five points defining the arcs on the right side of the egg and the green ones are the points needed to draw the left side. If we hide all circles and lines, replacing some of them with segments we get the picture below showing the five points needed for the right side of the egg.
**Task:** Use GeoGebra or some other dynamic geometry software to create a five-point egg. Try dragging the points to see how the shape of the egg changes. Note: if the egg is drawn correctly all the meeting point of different arcs should remain smooth after the dragging.

**Experimenting**

The bottom half of the five-point egg demonstrates a construction that can easily be continued, i.e. we can pick more points on the line segments and create a spiral-looking figure like:
This was done using only the line tool, the arc tool and the two point tools. We can drag all the blue points to change the construction. Drawing segments and hiding the lines and points we get the picture below:

![Spiral above with lines and labels hidden](image_url)

**Fig. 19** Spiral above with lines and labels hidden

*Task:* Create the spiral above. Experiment with colors to make a nicer picture.

*Task:* At [http://mathworld.wolfram.com/ThomsEggs.html](http://mathworld.wolfram.com/ThomsEggs.html) there are pictures of Thom’s eggs. Construct the two eggs using circles, arcs and lines.

*Task:* Search for “Golden Egg” on the internet to find a picture of the Golden egg and construct it using circles, arcs and lines (a picture can also be found in Mathographics).

*Task:* Experiment with similar constructions and make your own.

**Acknowledgment**

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**References**


How to add infinitely looong sums ...
How to add infinitely long sums ...

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Introduction

Addition belongs to basic arithmetic operations. This well known and relatively simple operation in case of adding two, three, four, etc. numbers becomes an unexpected problem in case of addition of infinite number of numbers (addends). This problem was for the first time encountered in the ancient Greece.

Let us mention two stories, known as Zeno’s paradoxes. In the first one, let us imagine an archer in a distance from the target he is aiming at. An arrow thus has to pass the whole distance $d$ before it reaches the target. It also means that an arrow has to pass the half of the distance, i.e. the distance $\frac{d}{2}$. Then it has to move through the half of the rest of the distance, which is $\frac{d}{4}$. From the remaining rest of the distance it has to pass at first again just the half, that is the distance $\frac{d}{8}$, etc. It seems as if the loosened arrow has to pass an exact remaining distance to its target all the time, in other words – it never reaches its target.

Similar point can be found in the story about Achilles and the tortoise. Achilles, one of the biggest warriors of Greek mythology, is in a footrace with the tortoise. His self-esteem allows the tortoise a head start, let us label this distance $\alpha$. After the start, Achilles runs, of course, faster, and without any problems passes the distance $\alpha$. However, the tortoise did not wait for him and meanwhile moved on the distance $\beta$. It will then take Achilles some further period of time to run that distance, let it be arbitrarily short, in which period the tortoise will advance another tiny distance $\gamma$ farther. Thus, he has to reach this third point, but whenever Achilles reaches somewhere the tortoise has been, he still has farther to go, because the tortoise moves ahead. We can observe that even a thoroughly chosen competitor can never overtake even the slowest tortoise.

So far the verbal analysis of these situations. However, reality and experience teach us that there would hardly be anyone facing the shooting – as a matter of fact that the bullet never reaches him/her. And maybe even less people would race with the tortoise; they are convinced that they can overreach it easily. Why then such an apparent contradiction appeared in the analysis of the stated situations?

The origin of our problems (or rather the problems of Classical Greek mathematicians and philosophers) probably lies in the fact that the motion of an arrow – a dynamic action – was split into few static situations. Nevertheless, how many such static situations shall we take to “put together” the whole dynamic process? After transforming this question into mathematics we realize that we need to add infinitely many numbers.

Human imagination in the mentioned times (and surely even today) refused to accept the fact that by adding infinitely many numbers (or more precisely, infinitely many positive
numbers) a finite product – a finite real number can be obtained. In order to illustrate the above mentioned fact, we will try to bring it closer using simple figures.

**Figures and sums**

Construct a square given a side \( a = 1 \). Divide the square into two identical rectangles, their areas are obviously equal \( \frac{1}{2} \). Choose one of them and divide it again to two identical parts – this time there are identical squares with the area \( \frac{1}{4} \). Pick up one of them and divide it into two identical rectangles with identical areas \( \frac{1}{8} \). Infinitely continuing in this process, the whole basic square with the area \( S = 1 \) will be filled (see Figure 1 a, b, c, d, e, f). Natural conclusion is thus the statement that

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1. \tag*{(*)}
\]

![Fig. 1 Numerical series](image)

Let us mention that, as usually in mathematics, there is quite convenient symbolism also in case of infinite numerical series representation, i.e.

\[
\sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n.
\]

If we look similarly at the case of a flying arrow, it can easily be proved that an arrow reaches the target in the distance \( d \) from the archer. For the sum of the sections, which an arrow crosses, applies

\[
\frac{d}{2} + \frac{d}{4} + \frac{d}{8} + \cdots + \frac{d}{2^n} = d \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots \right) = d.
\]
Let us try if the above stated result is obtained independently of the way how the given square is divided. For that purpose, construct again a square with a side $a = 1$. Divide it by a diagonal into two parts – two right triangles with equal areas $\frac{1}{2}$. Choose one of them and divide it by an altitude onto the hypotenuse into two other identical right triangles with the areas $\frac{1}{4}$. Divide one of them with an altitude onto the hypotenuse into another right triangles, this time their areas equal $\frac{1}{8}$. If continuing in this process to infinity (see Figures 2 a, b, c, d), by union of all the obtained triangles we gain the original square, therefore by adding areas of all the obtained right triangles we get the area of the given square. Thus, it can again be concluded that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots = 1.$$  

With slight changes these ideas could be transferred also into space, if we consider a cube with an edge $a = 1$. We can easily find two divisions of a cube equivalent to described divisions of a square, so that the volumes of obtained solids equal the volume of the whole cube. We would thus confirm that $\sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n = 1$. Let us add – this time without the detailed analysis – one more view of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots$. A square with a side $a = 1$ would be again a help. Find the midpoints of its sides. By their matching we obtain another square, and we continue in this process. Then it is from the Figure 3 clear that

$$\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots + \frac{1}{2^{n+2}} + \ldots = \frac{1}{4}.$$  

By adding the first two terms of the series we obtain the equality

$$\frac{1}{2} + \frac{1}{4} + \left( \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots + \frac{1}{2^{n+2}} + \ldots \right) = \frac{3}{4} + \frac{1}{4} = 1.$$  

Fig. 2 Numerical series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots$
Fig. 3 Numerical series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots$

Stated ideas suggest that the sum of the infinitely many positive addends really can be a finite real number. Let us prove that series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots$ does not have any “supreme” position – that it can be arrived at the same conclusion when considering many other infinite numerical series. We will use simple illustrations for better demonstration.

A square proved to be a very good device in the above example situations – construct therefore once again a square given a side $a = 1$. Divide it with two midsegments into four smaller squares, their areas equal $\frac{1}{4}$. Choose one of them and divide it in the same way into four identical smaller squares, their areas being $\frac{1}{16}$. As can be expected, again pick up one of these squares and divide it in the stated way into identical squares with areas equal $\frac{1}{64}$.

Repeat the division of a shape infinitely many times (see Figures 4 a, b, c, d, e, f), and observe only those squares that are colored in the figure. The sum of all their areas, i.e. the sum $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots + \frac{1}{4^n} + \ldots$, obviously substitutes one third of the whole area of the original square. Thus we get $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots + \frac{1}{4^n} + \ldots = \frac{1}{3}$. i.e. $\sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n = \frac{1}{3}$. 
Once more, do not trust everything you see, and let us try to prove the obtained result in a different way. We will stop referring to squares exclusively and try to use equilateral triangle this time (the length of the side does not make any difference), let its area be labeled with letter $S$. By matching its midpoints we obtain four identical equilateral triangles, areas of which obviously equal $\frac{S}{4}$. Choose one of them and with the help of midsegments create four identical equilateral triangles, their areas are $\frac{S}{16}$. After the third analogical step we get another quaternion of equilateral triangles, this time their areas being $\frac{S}{64}$. We can continue in this process infinitely (see Figures 5 a, b, c), gaining thus the set of triangles highlighted in color in our figure. Considering the area they comprise in the original triangle, it can be claimed that the equation $\frac{S}{4} + \frac{S}{16} + \frac{S}{64} + \ldots + \frac{S}{4^n} + \ldots = \frac{S}{3}$ is valid, resp. after the simplification by a nonzero number $S$ we gain equivalent equation $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots + \frac{1}{4^n} + \ldots = \frac{1}{3}$.
Up to now, the emphasis was put within highlighting the paradox that the sum of the infinitely many addends is a finite number – only on the sum of positive numbers. Further, let us think about more general problem: Can the sum of infinitely many positive and negative addends be a finite number? Let us approach this problem from the geometrical point of view, too.

At the beginning, let us choose the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \left(\frac{-1}{2}\right)^{n-1} + \cdots$ as we can make use of one of the previously listed figures while exploring this problem. We construct a square with a side $a = 1$ again and divide it in the way described in the figure 4. Despite that, this view is going to be different – while in the previous situation the colored shapes were observed, in this case the rest of the figure is going to be interesting for us. The description of this situation is included directly within the figures 6 a, b, c, d, e, f, g, h, i. It should be once again emphasized that this time the off-color part of the illustration is important. They after all build up the two-thirds part of the original square and consequently we can write

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \left(\frac{-1}{2}\right)^{n-1} + \cdots = \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{n-1} = \frac{2}{3}.$$
Fig. 6 Numerical series \(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + \left(-\frac{1}{2}\right)^{n-1} + \ldots\)
Let us add at least one similar sample, let us deal with the infinite numerical series
\[ 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{12} + \ldots + \left( -\frac{1}{3} \right)^{n-1} + \ldots. \] Although it is hard to believe, it is enough to use a square with the side \( a = 1 \) again to prove this situation. The division principle is understandable from the series of figures 7 a, b, c, d, e, f. It should be added that the attention should be paid to the off-color shapes obtained in the process. It is not hard to see that the infinite repetition of this process leads towards gaining the shape, the area of which equals three quarters of the whole square. That means that we obtain the equation
\[ 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{12} + \ldots + \left( -\frac{1}{3} \right)^{n-1} + \ldots = \sum_{n=1}^{\infty} \left( -\frac{1}{3} \right)^{n-1} = \frac{3}{4}. \]

**Fig. 7** Numerical series
\[ 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{12} + \ldots + \left( -\frac{1}{3} \right)^{n-1} + \ldots \]

Infinite numerical series which we dealt with can be generally recorded as
\[ q + q^2 + q^3 + \ldots + q^n + \ldots \] resp. \( \sum_{n=1}^{\infty} q^n \), and they are referred to as geometric series, number \( q \) is the constant multiple of the series. It should be mentioned that values of constant
multiple \( q \in (-1;1) \)** were not chosen by chance, as these are the cases when the sum of geometric series is finite. If the symbol \( s \) refers to the sum of series \( \sum_{n=1}^{\infty} q^n \), then we have already encountered the following situations: \( q = \frac{1}{2} \Rightarrow s = 1 \), \( q = \frac{1}{4} \Rightarrow s = \frac{1}{3} \), \( q = -\frac{1}{2} \Rightarrow s = \frac{2}{3} \), \( q = -\frac{1}{3} \Rightarrow s = \frac{3}{4} \). Unfortunately, it is possible that the sum of the geometric series is not dependant only on its constant multiple. Therefore, instead of the hypotheses about the form of the formula for the sum \( s \), let us try to discover the demanded formula with the help of geometrical sketches and reflections.

![Fig. 8 Numerical series \( 1 + q + q^2 + ... + q^n + ... \)](image)

At first, construct the rectangle CFED, whereby \(|CF| = 1, \ |CD| = q\). Construction of a point A is clear from the figure 8a, as well as point B emerges as an intersection of half-lines AE and CF. The similarity of the triangle CFE and all other colored triangles implies that the lengths of given horizontal sections equal \( q \), \( q^2 \), \( q^3 \), etc. From the similarity of triangles DEA and CBA we obtain the equation \( 1 + q + q^2 + ... + q^n + ... = \frac{1}{1-q} \).

It should also be noted that this result was gained thanks to the “completion” of a square MSON in the figure 8b and through the use of similarity of triangles NPO and MRN.

Thus a simple formula for the sum of the series \( \sum_{n=0}^{\infty} q^n \) was acquired. We admit that the situation was simplified thanks to the first term in the series, which is number 1. How it would like in the general formula, i.e. in case of series \( a + aq + aq^2 + ... + aq^n + ... \)?

** We also do not deal with the trivial case of \( q = 0 \).
Observe figure 9a. Construct straight lines \( y = x \) and \( y = qx + a \) in the system of coordinates. The second line intersects an axis \( o_y \) in the \( a \) distance from the origin of coordinates. The length of the segment led from the intersection parallel to the axis \( o_x \) up to the intersection with the straight line \( y = x \) equals also \( a \), it is the axis of a quadrant. Length of another vertical segment is \( aq \), it is the value of a function \( y = qx + a \) in the point \( a \). Further procedure of the construction of colored parts is obvious. However, both coordinates of the point of intersection of the given lines correspond the sum of the individual sections, wherefrom the equation \( a + aq + aq^2 + ... + aq^n + ... = \frac{a}{1-q} \).

Analogically, in figure 9b the addition of terms of series \( a - aq + aq^2 - ... + a(-q)^{n-1} + ... \) using lines \( y = x \) and \( y = -qx + a \) is illustrated.

It appears that the sum of infinite geometric series is dependant on the first term of the series as well as on its constant variable. Finally, let us verify the obtained formula by a simple calculation. Let us assume that the sum of geometric series \( \sum_{n=0}^{\infty} aq^n \) equals value \( s \), i.e. \( \sum_{n=0}^{\infty} aq^n = s \). That means that the equation \( s = a + aq + aq^2 + ... + aq^n + ... \) is valid. Multiply the equation by a nonzero number \( q \), so that we get \( sq = aq + aq^2 + aq^3 + ... + aq^{n+1} + ... \). By subtracting these equalities we obtain \( s - sq = a \), wherefrom \( s = \frac{a}{1-q} \), which is the formula derived from the figures.

**Conclusion**

Infinite number series have long been considered non-trivial mathematical issues. Obviously, addition of infinitely many numbers cannot be an easy task. Infinite number series used to be
ascribed mythical properties. Luigi Guido Grandi (1671 – 1742) presented notations
\[
\sum_{n=1}^{\infty} (-1)^{n+1} = (1-1) + (1-1) + ... = 0 \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^n = 1 + (-1+1) + (-1+1) + ... = 1
\]
as the evidence of the God’s existence; the notations show the birth of “something” from “nothing”. At study of such demanding topic it seems reasonable to apply illustrative pictures as an aid in order to capture the dynamical progression of terms of partial sum sequences. Moreover, the illustrations allow deducing the result of the process directly from the picture. The contribution discusses selected cases and shows the generalization for geometrical series.

Acknowledgment

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References

Creativity in Geometry on the Playground
Creativity in Geometry on the Playground

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Introduction
Supporting of positive students’ attitudes towards mathematics is connected with their achievements in school education. Researchers concluded that positive attitude towards mathematics leads students towards success in mathematics. Attempt to improve attitude towards mathematics at lower level provides base for higher studies in mathematics. It also causes effect in achievement of mathematics at secondary school level (Ma & Xu, 2004). According to Haladyna et al. (1983), the general attitude of the class towards mathematics is related to the quality of the teaching and to the social-psychological climate of the class. Work in the group when solving any problem or task in mathematics can support positive climate in educational process.

Spatial concepts understanding
One part of mathematics is geometry and students’ ability to find and see geometric objects in their everyday environment is reflected in their creativity when posing their own geometrical problems. Perception of plane and space also influences students’ achievements in mathematics. The results of study of Guzel and Sener (2009) show that spatial ability (three-dimensional thinking) improves students’ understanding of symbols, shapes, tables, and figures. Besides, it assists students in comprehending drawings easily, commenting the visualized information, creating contexts among different concepts easily, generalizing complex concepts, and thinking in different ways. Accordingly, spatial ability plays crucial role to be successful in mathematics, specifically in geometry, for the reason that the field is based on visualization. According to the theory of van Hiele (1999), there are five levels of understanding spatial concepts through which children move sequentially on their way to geometric thinking: visualization, analysis, abstraction, deduction and rigor.

Level 1 (Visualization): students recognize figures by appearance alone; often by comparing them to a known prototype, the properties of a figure are not perceived; students make decisions based on perception, not reasoning.

Level 2 (Analysis): students see figures as collections of properties; can recognize and name properties of geometric figures, but they do not see relationships between these properties; when describing an object, a students might list all the properties they knows, but not discern which properties are necessary and which are sufficient to describe the object.

Level 3 (Abstraction): students perceive relationships between properties and between figures; can create meaningful definitions and give informal arguments to justify their reasoning; logical implications and class inclusions, such as squares being a type of rectangle, are understood; the role and significance of formal deduction, however, is not understood.
Level 4 (Deduction): students can construct proofs; understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions; students should be able to construct proofs such as those typically found in a high school geometry class.

Level 5 (Rigor): students understand the formal aspects of deduction, such as establishing and comparing mathematical systems; can understand the use of indirect proof and proof by contrapositive, and can understand non-Euclidean systems (cited in Mason, 1999).

Van de Walle (2001) mentioned four characteristics of these levels of thought:

- The Van Hiele’s levels of geometric reasoning are sequential. Students must pass through all prior levels to arrive at any specific level.
- These levels are not age-dependent in the way Piaget described development.
- Geometric experiences have the greatest influence on advancement through the levels.
- Instruction and language at a level higher than the level of the student may inhibit learning.

Goldenberg, Cuoco and Mark (1997) state that geometry is the ideal vehicle for building the ‘habits of mind’. In geometry learners can search for invariants (patterns) and use these to arrive at generalizations, experiment, analyze, synthesize, visualize, describe and give proofs for their conjectures. Unfortunately geometry is a neglected field. Curricula tend to present a visually impoverished mathematics.

Hutchinson (1971) states that all the abilities of all children must be developed fully if education is to do its job. Recent research, however, points to the fact that some important abilities are not being developed by current classroom methods. Thinking and creativity are some of the neglected abilities in the classroom.

Creativity

Creativity is an essential feature of personality which each of uses in our everyday life, since it allows us flexibility when dealing with real life situations. Study of mathematics should be seen as one of the opportunities for its development, although creativity is not traditionally associated with maths. (Švecová, Rumanová, 2012)

Mathematics can be an interesting activity not only for the mathematicians but also for the teachers and students. (Ponte, 2008) People who think independently feel the need to make sense of everything based on personal observation and experiences rather than on information they were given without questioning it. It is more tempting to concentrate just on the memorization of facts and practice algorithmic skills, than to work on creativity and independence in thinking. (Hoffmann, 2008)

According to survey conducted by Blažková and Vaňurová (2011), creativity of children depends in great deal on teacher’s approach. When pupils solve only classical tasks by always using the same methods, they have problems to change their learned way or create a task independently. Children can develop certain commodity in thinking, little initiative or even unwillingness to work.

Creativity is defined as a production of new and original ideas. (Zelina, 1996) Research on creativity, at the group level, has highlighted the potential trade-off between social control and creativity (Nemeth & Staw, 1989).
Creativity in the mathematics classroom is not just about what pupils do but also about what teachers do. If teachers think creatively about mathematical experiences that offer to their pupils they can open up opportunities for pupils to be creative.

Mathematics is as much about problem posing as it is about problem solving. It is about noticing within a situation that there is a question waiting to be asked. At this point, creativity lies in noticing that there is something to be investigated. When setting up situations in the classroom teachers should make an effort to choose contexts that offer students opportunities to pose their own problems.

Creative teaching requires from teacher to create and exercise such tasks, which would enable pupils to use their acquired knowledge more freely, in new contexts and when solving new and unknown problems. (Lokšová & Lokša, 1999) When talking about creative activity, just as about any school activities, what children produce depends not only on their own abilities but also teacher’s skills. Teacher who has wide spectrum of interests and who wants to share them with children in the school and outside it supports creativity of pupils more than stereotypical teachers (Vidermanová, Melušová, 2011).

**Some ideas how to support creativity in geometry**

Following problems could be perceive more for motivation as the ideas for activities which lead to creation mathematical problems and take a place for solver’s own creativity. The realistic content and real environment are important in following ideas for the activities.

In the first phase of the activity students can looking for “any geometry” outdoors, such on playgrounds and make photos with interesting object when walking in the playground. Creation of some ideas for geometrical problems which are based on specific pictures is the next phase. There are no specific measure related to geometrical objects in the problems, they are open problems which could be completed and solved by students.

The main goal during creation the problems is to develop student’s dynamical mathematical thinking and creation of mathematical tasks. For creation task, students can fill measure in figure:

- by estimating according to their own experiences,
- by approximating to similar real objects,
- by measuring concrete picture and adapting it in appropriate gauge,
- by searching similar object in their environment and measure it.
Problem 1

There is a tree stump in the picture shaped as oblique cylinder.

- What could be the volume of tree bark which has peeled off of this stump, if its thickness was 8% of its radius?
- What could be the maximum distance between ground and the ant which is walking on the tree stump?
- What could be the weight of sawdust which we could get from this whole stump, if its wood density is 690kg/m$^3$?

Problem 2

There is a plinth for decorative pillar shaped as regular 6-sides’ prism.

- How many litters of concrete are possible to pour into the pillar?
- How many small stones were needed for decorating exterior ...., if average area covered by one stone is 7cm$^2$?
- In how many ways can the stand be divided into two equal parts?

Problem 3

The figure shows eleven border cylindrical columns.

- How many kg of red and white painting was necessary for these columns, if 1 kg paint will cover approximately 8 m$^2$ area?
- How long should be the string that we would use to connect the poles, if we make a loop around each of them? Tying the rope at the beginning and end we can count 1.2 m.
Problem 4

There is a children's roller treadmill in the picture.

- How many wooden slats were needed for its preparation and how many rivets to attach them?

*The correctness of solutions can be verified in this picture.*

- If you would like the yellow part of the cylindrical rods with a length equal to the diameter of the cylinder running repainted into green, how much area would be green?

Problem 5

The picture shows the signs placed at the beginning of the playground.

- What are the different geometrical shapes in the picture?

- Redraw two units, which are in the middle of the sign into the square grid. The length of the boxes should be the same length as the distance between two quarters of the circles. Sides of the rectangle are in ratio of 1:4 and the length of the shorter sides of the rectangle would be half the length of the box.

- What is the ratio between black and white areas?
Problem 6

The pillar in the picture is on a pier at the pond and serves to attach the boat.

- How tall can be the pillar, if we know the length of its shadow?
- What would be the length of a rope you would consolidate over the poles and wooden panels of the pier, with a rope to a pillar of any angle?
- Pillar, its shadow and the rope stretched out between them, could form a triangle. What length of a sides and the size of the internal angles should have newly formed triangle?
- Pier has rectangular wooden slats covered in two rows, as in the picture. How many wooden slats would be needed to cover the pier, if one slat represents 3% of the whole pier?

Problem 7

There is one of climbing frames in the playground shaped as a half-cylinder in the picture.

- What would be the shape of a climbing frames developed into the plane, if there are a square holes?
- How many meters of a pipe would be needed for the construction of the climbing frame? The diameter of half-cylinder should be equal as its height.
- How many kilograms of yellow and red colour do we need for coating bars of climbing frame if we know a radius of a pipe?
Problem 8

There is a car wheel with a disc in the picture.

- How many different straight lines could we use to divide the disc into two identical parts?
- How many percent of wheel represents a tyre?
- What would be the width of the tyre, if it was three times the diameter of the tyre?

Problem 9

There is a rectangular trapezoid, created from pieces of tree trunk in the picture.

- How much soil would we need if we wanted to use the trapezoid as a decorative planter?
- Trapezoid formed this way could be used as a flowerbed. How many flowers would we need to plant if the distance between flowers was 20 cm?
- What would be the size of the internal angles of trapezoid if the bases were in the ratio 1:3?

Problem 10

There is a rubbish bin which shaped as perpendicular prism in the picture.

- What can be the maximum volume of rubbish in the full bin that not hangs over the edge of bin?
- How much more wood would we need for lining bin, if we wanted to dado it without spaces?
- If we empty one-fifth of the bin the total weight will be 80 percent of the full one. What is the weight of the empty bin?
Using ICT technologies when solving the problems

For more than forty years, innovative educators have been optimistic about computer uses in schools. Most analyses of ICT in the educational sector focus on the impact it has had on pupil learning. (Carnoy, 2004)

Ittigson & Zewe (2003) cited that technology is essential in teaching and learning mathematics. ICT improves the way mathematics should be taught and enhances student understanding of basic concepts.

The use of ICT in teaching mathematics can make the teaching process more effective as well as enhance the students’ capabilities in understanding basic concepts. Nevertheless, implementing its use in teaching is not without problems as numerous barriers may arise. Some of these problems define Keong, Horani, Daniel (2005).

Kalaš (2001) define ICT as follows: The term ICT understand computing and communication tools that aid in teaching, learning and education, at work and in life in general. These include computer, Internet, e-mail, mobile phone, calculator, electronic diaries, etc.

In the next section we show the possibility of using certain ICT in solving math problems.

Solving the problems with using software GeoGebra

One of the most appropriate and available software for school education in recent years is GeoGebra. GeoGebra is a learning tool for math that is used all over the world. It’s a free geometry, algebra and calculus application, created for teachers and students from elementary school to university level - See more at: http://www.itslearning.net/geogebra-makes-math-more-fun#sthash.SkQvoBuH.dpuf.

In the following examples we want to highlight the dynamic elements of GeoGebra that can affect the outcome or the number of solutions of problem. Two specific problems connected with the photos above and their solutions demonstrate the use of software GeoGebra.
Problem 11 (connected with problem 5)

The picture shows the signs placed at the beginning of the playground.

Redraw two units which are in the middle of the sign into the square grid. The length of the boxes should be the same length as the distance between two quarters of the circles. Sides of the rectangle are in the ratio and the length of the shorter sides of the rectangle would be half the length of the box.

With using the GeoGebra we repainted quadrant and a rectangle by entering into the selected grid. One of the advantages of the software is the ability to play a given construction and thereby advance to rethink the didactic process work with students. We can also control the solutions and show the work with software in the constructions steps.

Fig. 1. The road sign redrawn in a grid
Problem 12 (connected with problem 6)

The pillar in the picture is on a pier at the pond and serves to attach the boat. Pillar, its shadow and the rope that we could stretch out between them, form a triangle.

- How does the length of the rope depending on the length of the shadow? When does the length of the shadow reach its maximum and its minimum?
- How does the length of the rope depend on its attachment to the pillar?

In the solution of first part of a task, we use the property that the length of the shadow depends on the impact of sunlight on the pillar. As we can see in the figure 2, this situation can be modelled in GeoGebra as the movement of point $S$ (which represents the sun) on a circle which represents the path of the sun on the sky. Situation in the figure 2 is only a model in which the shadow position is neglected and we focus to its length. This allows us to simulate three-dimensional effect in the plane, while the radius of the earth and the sun are also neglected.

Fig. 2 The dependence of a shadow on the position of the sun
In the second part of a task, the length of a rope depends on its location on the pillar. In GeoGebra (Fig.3), the length of a rope can be changed by the movement of point X on the segment AB which represents the pillar. By placing the point X to point B, we get the maximum length of a rope. When placing the point X to point A, a rope length is equal to the shadow length.

![Fig.3 The dependence of a rope length on its location on the pillar](image)

**Creation of problems with using tablet**

New technologies, as tablets, can be a powerful tool for learning and comprehension in educational process in the classroom. The interactivity can provide a very engaging experience, definitely for elementary school aged students. Tablets not only make technology accessible to young children, but also research suggests that tablets and apps can improve learning. Tablet PCs can create an environment that can maximise student learning opportunities, empowering both student and teacher. If used to its full potential it captures clear and recordable mathematical thinking in action and can provide purposeful and timely feedback.

According to the statements of teachers who implemented tablets to their lessons, tablets can help to keep student’s attention. The using of specific tablet applications leads to increasing of motivation and students’ interest of education. We can found a lot of various apps for teaching mathematics, which can be also used for solving and posing mathematical problems.

For instance, application for tablets which allows us to measure the distance and the height of objects – climbing frames on the playground, can be chosen. The photos made by tablet and obtained information and materials are appropriate for creating mathematical problems. This is one
of the possibilities how to develop creativity and geometric skills of students. There is one example in the Fig. 4.

Example 1

The real situation is transformed and constructed in a plane picture and measured values are used for designing the geometric problems. Pythagorean Theorem and trigonometric functions are the topics of curriculum where these problems can be implemented.

![Fig.4 The photo of tablet application 1](image)

Here are some ideas for creation and solution the problems:

- How far is the top of climbing frame from the pupil’s eyes? According to the Fig.5, Pythagorean Theorem can be used for calculation the distance $x = \|SR\|$, the distance between the eyes and the top of climbing frame.
- Under what angle pupil see the base of climbing frame? Trigonometric functions (tg, ctg) can be applied in counting the angle $PCQ = \alpha$ (Fig. 5).
• How tall must be a pupil that moves about 2 meters toward the climbing frame and the angle $\alpha$ is the same? Except the Pythagorean Theorem, the similarity between the triangles $PQS$ and $TQO$ can be used for counting the length $y = OT$ that represents the height of a pupil (Fig. 5).

Example 2

When the height of chosen blue climbing frame is measured, tablet application for measuring the distance from the object according to its height can be used. The distance between measured object and the pupil with tablet in his/her hands is 17.3 m in the picture (Fig. 6).

![Fig.6 The photo of tablet application 2](image)

Geometrical model of this situation is in the Fig. 7. The problems focused for counting the angles or length, distance, as in the example 1 should be also created. Another possibility is to use tablet application for counting the steps and to compare children and adult steps. Number of steps needed to pass a given distance or comparing the real and measured distance can be also realized.

![Fig.7 Geometrical model of the measuring 2](image)
There are some other photos from the playground as an inspiration for posing new mathematical problems below.

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Geometric interpretation of special theory of relativity
Geometric interpretation of special theory of relativity

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Introduction

The theory of relativity is a beautiful physical theory, which brings the new point of view to the physics for our Universe through Albert Einstein. For ordinary people it can look strange and surprising, because it does not mirror our daily experience in our slowly moving world. The theory shows the phenomena and events close to a speed of light. At the beginning a lot of physicists try to show, that the theory is wrong. They find out a lot of paradoxes, which would contradict the results of the theory. But in fact, all paradoxes could be explained by this theory.

The solution of these paradoxes can be mathematically difficult for readers with general knowledge of the calculus; however it could be very interesting and unexpected. Also there is the other way – geometric interpretation, which can be used for explaining the paradoxes. There are Loedel, Brehm and Minkowski diagrams.

Students who are still did not meet in the context of learning process with STR usually know Albert Einstein and his probably most famous photograph (Fig. 1). Most of them can also say $E = mc^2$, but explain what this equation means, they cannot know.

In this publication we present Loedel diagrams in this publication which use basic geometry and trigonometry. It is appropriate tool to solve different situations, problems and paradoxes. With the LD student can focus purely on the physical nature of the problem and directly see the effects of TR. On the other hand, LD also allows the derivation of basic algebraic relations TR, directly from diagrams without complicated mathematical apparatus.
1 Loedel diagram

In 1948 a simple geometric procedure was suggested by Enrique Loedel Palombo in Einstein theory. The Loedel diagram was based on the invariant interval. Our approach is to show geometry of Loedel diagram and after derive from it all consequences.

Geometry of LD

The Loedel diagram describes two observers in the two frames. Every observer in own frame is in the rest while the frames are moved to each other. It is built by two two-dimension diagrams, which represent one direction space axis $x$ resp. $x'$ and the time axis $ct$ resp. $ct'$ for both frames $\mathcal{K}$ and $\mathcal{K}'$. The main condition for the Loedel diagram is $x' \perp ct$ a $x \perp ct'$ (Fig. 2).

![Fig. 2 Loedel diagram](image)

The coordinates of the points, which are called “events”, in Loedel diagram are made by parallels. Every event has the same time coordinate on the parallel which is paralleled with axis $x$ resp. $x'$. Every event has the same space coordinate on the parallel which is paralleled with axis $ct$ resp. $ct'$ (Fig. 3).

![Fig. 3 Loedel diagram](image)

$^1$c is speed of light in vacuum, which is the fundamental constant for all observer. The meaning of $ct$ is time in unit of meter - 1meter is the time which light in the vacuum flew 1m distance.
Fig. 3 Events in the same time coordinates or same space coordinates in frame $K$ and $K'$.

The time coordinate $ct, ct'$ resp. $ct', ct$ of event $A$ is given by intersection of parallel with axis $x$ resp. $x'$ through event $A$ and axis $ct$ resp. $ct'$ in the frame $K$ resp. $K'$ and space coordinates $x, ct$ resp. $x', ct'$ is given by intersection of parallel with axis $ct$ resp. $ct'$ through event $A$ and axis $x$ resp. $x'$ in the frame $K$ resp. $K'$. In the frame $K$ the event $A$ has coordinates $(x, ct)$ and in the frame $K'$ has coordinates $(x', ct')$ (Fig. 4).

Fig. 4 Coordinates of event $A$

**Velocity between frames**

The angle between $x$ and $x'$ is the same as between $ct$ and $ct'$. What does the angle between $ct$ and $ct'$ mean?

For example, we have two observers, which one is moving away from other with the velocity $v$. The observer in the frame $K'$ switches the light on as he passes the observer in frame $K$, point $S$. 

Fig. 5
In the time $ct_A$ the observer switches the light off, event $A$. For observer in frame $K$ the light is moving away with velocity $v$ to distance $x_{A'}$.

![Fig. 6 Velocity between frames](image)

From the grey triangle in LD can be seen that

$$\sin \alpha = \frac{x_{A'}}{ct_{A'}} = \frac{v}{c}.$$  

The velocity between frames in relative of speed of light is given with sinus of angle $\alpha$.

The maximum value which can be obtained is 1. It means that maximum speed is the speed of light.

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**The speed of light is maximal possible speed.**

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**The Construction of the Loedel diagram in Geogebra**

Geogebra is the geometrical software, which can be used for solving problems in Loedel diagrams. Open the new document in the Geogebra. At beginning, it is necessary to show how to construct the LD.

Choose the tool *Line* and click on it. Select two points for create the horizontal line $a$. Right-click on the created line and on the pop up menu choose *Rename* to rename it as “xAxis”\(^2\). If you are not satisfied with orientation or position of the axis, you can change it. Select the tool *move* and pick the one of the point A or B and drag it to new position. We can change the line style of the created axis to dot chain line. Right-click to “xAxis” and in pop up menu you can choose *Object properties*. In the opened window *Reference*, you can click on the label Style and you can change the line style to dot chain. If you would like to change the colour, click on the label Colour and pick the black colour.

\(^2\) because you cannot use letter “x”
clicking to pallet. Now we have done the $x$ axis of the frame $\mathcal{K}$. With tool *perpendicular line* we can set up the $ct'$ axis. Click on the *perpendicular line* button and select the line “$xAxis$” and point A. You create the perpendicular line “$a$” and rename it as “$ct'Axis$”. Change the colour to red and line style to dot chain. Now we have done the $ct'$ axis of the frame $\mathcal{K}'$.

Now we are going to create the $x'$ axis which is rotated to $x$ axis above the angle $\alpha$. We have to create a slider that will set the value of the angle $\alpha$. Then we connect this slider to angle between axis $x$ and axis $x'$. We select the tool *slider* and click to the Graphics windows. The slider properties window appears. In this window, choose angle radio button. Leave name unchanged $\alpha$ and interval set from $0^\circ$ to $90^\circ$ with the increment 1°. If you are done, click the button *apply* and slider is created. Now you click on the *Angle* and in pop up menu choose *Angle with Given Size*. First, select point B and after A. Then in the window which appears write $\alpha$ or click to the input line. The button with a sign $\alpha$ appears in the right corner. Click on it and select the letter $\alpha$. Click the button O.K.. Scrolling the slider is changing the value of angle. Create the line by selecting the point A and B’. Rename it as “$x'Axis$” and change colour to red and line style to dot chain. Finally you have to create the $ct$ axis which is perpendicular to $x'$. Choose the *Perpendicular Line* and select the “$x'Axis$” and the point A. Rename it to “$ctAxis$” and change colour to black and line style to dot chain.

Now you have finished Loedel diagram and you should save it before using.
**Word-line**
The trajectory of object in spacetime will be called **world-line**. In LD is defined by curve, which shows the space and time coordinates in every point. When point A is in the rest in the frame \( \mathcal{K} \), the word-line is parallel to axis \( ct \). When point B is in the rest in the frame \( \mathcal{K}' \), the word-line is parallel to axis \( ct' \) (Fig. 7).

**Fig. 7 Word-line**

**Contraction of Length**

Our observer in frame \( \mathcal{K} \) catches the fish with length \( l \). How does the observer in moving frame \( \mathcal{K}' \) measure the length of the fish? He has to measure the position of one end A and other end B in the same time in the own frame (fig. 8).

**Fig. 8**
We have to construct the word-lines of the both ends of the fish. The length of the fish in $\mathcal{K}^\prime$ is defined by intersection of the world-line and an axis of simultaneity. We can choose the axis $x^\prime$. From the right triangle is apparent in LD in Fig. 9

$$\cos \alpha = \frac{l^\prime}{l} \quad \Rightarrow \quad l^\prime = l \cos \alpha = l \sqrt{1 - \sin^2 \alpha},$$

$$l^\prime = l \sqrt{1 - \frac{v^2}{c^2}}.$$

Then we can see not only from LD, but also from equation of contraction of length; the length of moving object is shorter for the observer in the rest as for the observer who is moving with object. The shortening depends on the velocity of moving object.

---

Moving objects are seen shorter than these in the rest.

---

**Task 1:** How will be the diagram changed, when the fish is in the rest in the red frame $\mathcal{K}^\prime$? Find the formula for this case (incident)?

**Task 2:** How fast would 12 m long bus move to be seen about 1 m shorter? (In LD you can use Thales’ theorem.)
Time dilation

At night, when we look at sky we can see a lot of flashing objects – planes. If the light of the plane flashes every second (Fig. 10) in the plane frame is it seen by observer on the ground every second too?

![Fig. 10](image)

We can say that one flash is an event \( \mathcal{A} \) and next one is the event \( \mathcal{B} \). In LD we can point this event to the rest in the frame \( \mathcal{K}' \). Time separation for this event is \( ct' \). These events have to be on the world-line of plane. We can find the coordinates of these events in the frame \( \mathcal{K} \) with parallel with axis \( x \) through event \( \mathcal{A} \) and event \( \mathcal{B} \). From the grey right triangle is apparent in LD in Fig. 11.

![Fig. 11](image)

\[
\cos \alpha = \frac{ct'}{ct} \quad \rightarrow \quad ct = \frac{ct'}{\cos \alpha} = \frac{ct'}{\sqrt{1 - \sin^2 \alpha}},
\]

\[
\tau = \frac{\tau'}{\sqrt{1 - \frac{v^2}{c^2}}},
\]

*Time in the moving frame runs slower for observer in the rest*
It is seen that time of moving system appears slower for the observer in the rest. For example the observer’s clock in the spaceship is moving with velocity $0.9c$ from the Earth. Every minute on the Earth takes in his point of view $2.29$ minutes.

Task 1: How is diagram changed, when flashing light has been in rest in the black frame? Find the formula for this case.

Task 2: An observer on the Earth has the peripheral speed $465$ m/s. The space station ISS is $400$ km about the Earth’s surface and their peripheral speed is $7.66$ km/s. What is the time difference for one day (24h) at the space station ISS and at the control center on the Earth?

Addition of the velocities
How do different observers perceive a moving object? Our experience shows that if you ride on the bicycle with speed $20$ km/h and the car overtake you with speed $30$ km/h. For the observer who stands on the road, the car is moving with speed $50$ km/h. Is the speed composition same for fast moving object?

Space ship moving away from the Earth with speed $v$, immediately after start it sends a probe in direction of movement of space ship. The speed of probe is $u'$ in the frame of space ship. What is the probe’s speed in the Earth frame?

Look at this situation in the LD (Fig. 12). First we have to draw the world-line of probe. We can do it because the speed of probe is known. In the spaceship (red) frame the probe travels for time $c\tau'$ distance $x' = u't'$. Call this event $A$ with coordinates $A(u'\tau'; c\tau')$. The line which connects the point $A$ and $S^3$ is world-line of probe. Now we only need to find the coordinates in the Earth frame because the speed of the probe is equal to distance which can travel for unit of time. We have to express the distance and time from LD in the Earth frame. We can find it from gray and blue triangle

---

3 The pint $S$ we define as the beginning of the coordinates systems LD. In that point the probe was sent from the spaceship.
Substituting to the equation for speed we obtain the formula for addition of the velocities.

\[ u = \frac{x_{\mathcal{A}}}{t_{\mathcal{A}}} = \frac{u' + c \sin \alpha}{1 + \frac{u'c}{c^2} \sin \alpha} \]
\[ ct_{\mathcal{A}} = \frac{ct' + u' \sin \alpha}{\cos \alpha} \]

\[ x_{\mathcal{A}} = \frac{u' + c \sin \alpha}{\cos \alpha}; \quad ct_{\mathcal{A}} = \frac{ct' + u' \sin \alpha}{\cos \alpha}. \]

Formula for the addition of the velocities is different than in classic physics. The velocity between frame is \( v \). If the speed of the object is \( u' \) for one observer, than the speed for observer in the other frame is given by formula

\[ u = \frac{u' + v}{1 + \frac{u'v}{c^2}} \]

where \( \sin \alpha = \frac{v}{c} \).

Task 1: How will the LD changed if the probe is sent in opposite direction? How will be the formula changed.

Task 2: Astronomers who travel from the Earth to Jupiter with speed 0.3c are sent the electromagnetic pulse to control center on the Earth. The speed of the electromagnetic pulses is speed of light. How fast is the pulse moving in the Earth frame?

Light signal in LD

An observer in the red frame \( \mathcal{K}' \), who is moving with the speed \( v \) and holding the shining light bulb. The speed of the light signal in the moving frame \( \mathcal{K}' \) is \( c \) (Fig. 13). We can construct the world-line of light signal in LD. The distance when the light can travel in the time \( \tau \) is \( x'_{\mathcal{A}} = ct'_{\mathcal{A}} \). After we construct it in the LD we can see that the light signal is angle bisector of the axis \( x' \) and \( ct' \), but also is the bisector between \( x \) and \( ct \). That means that this is the world-line of the light signal in the frame \( \mathcal{K} \).
The velocity of light signal is same for every frame and equals $c$.

Task 1: Construct the situation in LD, where spaceship is moving away from the Earth and sends the light signal back to the Earth.

**Relativity of simultaneously in LD**

As we have already showed at the time dilation and length contraction the distance and time may not be the same for two different observers. Even events which we observe can be seen otherwise. Relativity of simultaneously says that what is simultaneous in one system may not be simultaneous in another and vice versa.

Let's take a simple example. We have two observers one on the train passing through the platform and the other watching him at the platform. An observer travelling on the train is standing in the middle of the wagon and there is the lamp over his head. Both observers agreed that at the moment they passed each other the observer on the train switched the light on. They will observe when the light signal hits the beginning and the end of the wagon. Construct this situation in the Loedel diagram (system $\mathcal{K}$ is black, system $\mathcal{K}'$ is red).

Look at the situation at the moment when the observer on the train and the observer at the platform are passing (Fig. 14). Let the length of the wagon in rest is $2l'$ in the red system. We need to find world-lines of the beginning Z and the end K of the wagon. The world-line is parallel with the axis $ct'$ through these points. From the middle of the wagon, where the observer stands (point S) were sent the light signals in the direction and opposite direction of the wagon movement. The world-lines of the light signals is bisector of angle between the axis $x'$ and $ct'$ in the direction of wagon movement and bisector of angle between the axis $ct$ and $x$ in the opposite direction of wagon movement.

The intersection between world-line of the light signal and the world-line of the beginning of the wagon determines the event $A$ when the light signal hits the beginning of the wagon, respectively event $B$, when the light signal hits the end of the wagon. By determining the time coordinates of these events in both systems, we can say that for the observer on the train it happened simultaneously, but for the observer at the platform at first the light signal hit the end of a wagon and one moment later hit the beginning of the wagon.
Simultaneous of event is relative and depends on the movement of observers.

Task 1: Find the formula for time difference between events $A$ and $B$ for the observer at the platform, if you know the length of the wagon $l$ at the platform frame and speed of the wagon.

Task 2: Construct the LD in case, when the wagon is in rest next to the platform and the observer is moving on the trolley with speed $v$ at the platform.

Lorentz transformation in LD
Let us take any event $A$ in Loedel diagram. It could be impact of the asteroid to the Mars. On the Earth they observe that event and record the exact coordinates. The spaceship which is close to the Earth moves from the Earth with speed $v$. What are the coordinates of the event observed in the space ship?
We have to express the coordinates $x'_{\mathcal{A}}$ and $ct'_{\mathcal{A}}$ from LD using $ct_{\mathcal{A}}$ and $x_{\mathcal{A}}$. From the LD we can obtain using coloured right triangles for coordinates $ct_{\mathcal{A}}$ and $x_{\mathcal{A}}$ equations

$$ct_{\mathcal{A}} = x_{\mathcal{A}} \sin \alpha + ct'_{\mathcal{A}} \cos \alpha,$$

$$x_{\mathcal{A}} = ct_{\mathcal{A}} \sin \alpha + x'_{\mathcal{A}} \cos \alpha.$$  

Finding $ct'_{\mathcal{A}}$ a $x'_{\mathcal{A}}$ and substituting $\sin \alpha = \frac{v}{c}$, we obtain transformation formulas between frame $\mathcal{K}'$ and frame $\mathcal{K}$.

$$t'_{\mathcal{A}} = \frac{t_{\mathcal{A}} - \frac{x_{\mathcal{A}} v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x'_{\mathcal{A}} = \frac{x_{\mathcal{A}} - vt_{\mathcal{A}}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

*This transformation is called Lorentz transformation. The space coordinates are transformed only in direction of motion, not in perpendicular direction. That means $y'_{\mathcal{A}} = y_{\mathcal{A}}$ a $z'_{\mathcal{A}} = z_{\mathcal{A}}$.*

Task 1: Find an inverse Lorentz transformation from LD (express $x_{\mathcal{A}}$ and $t_{\mathcal{A}}$ using $ct'_{\mathcal{A}}$ and $x'_{\mathcal{A}}$).

Task 2: Imagine two parallel mirrors and the light signal is reflected perpendicularly between. The distance between mirrors is $d$. For observer in the mirrors frame it can look like a light clock. What does other observer see in the moving frame which is approaching perpendicularly to mirrors with speed $v$? Solve in LD.

The Paradoxes of the STR

It is interesting that STR is full of the paradoxes; however the most of the paradoxes comes from not understanding of the principles and consequences of the STR. We will show here some of these paradoxes, where is nothing paradoxical.

The Car and the Garage

We have the car we want to park in the garage, but the car is longer than the garage. There are gate from both side of garage. The front gate will immediately open when the front part of the car is going to pass the front gate and it will immediately close when the back part of the car is passing the front gate. The back gate is working the same way.

The driver can see the garage shorter because of the contraction of the length. He observes that the gates are open simultaneously and the whole car hasn’t been in the garage. For observer in the garage it happens little bit different. He sees that the car is shorter than the garage and the car has been inside the garage and the both gates were closed simultaneously. Is this the paradox?
When we look at the LD of this situation (Fig. 16) at first we have to define the frame of the garage and also the car. Let the garage be in the rest in the black frame $\mathcal{K}$ and the car is in the rest in the red frame $\mathcal{K}'$. The length of the garage is defined by distance of the front and back gates in black frame $\mathcal{K}$. The length of the car is defined by the distance of the front part and back part of the car in the red frame $\mathcal{K}'$. The world-lines of the front and the back gates are parallel with axis $ct$ and the world-lines of the front and back parts of the car are parallel with $ct'$. Intersection of world-lines define events, where and when the car is entering the garage and coming out from the garage.

From LD we can see the time order of events in the red frame $\mathcal{K}'$ - driver frame. Event 1 – the front part of the car enters the garage, event 2 – the front part of the car comes out of the garage, event 3 – the back part of the car enters the garage, event 4 – the back part of the car comes out of the garage.

For other observer in the garage - black frame $\mathcal{K}$ is the time order of events different. Event 1 – the front part of the car enters the garage, event 3 – the back part of the car enters the garage, event 2 – the front part of the car comes out of the garage, event 4 – the back part of the car comes out of the garage.

So, both observers have seen what they should have seen – they were looking at the same events, but the measurement of the space-time coordinates depends on the frame. So both observers were right.

Task 1: There is a colour bomb fixed on the car roof which is going to explode when the car is in the garage and the gates will be closed. Does the bomb explode?

Task 2: What happens if the color bomb is fixed on the roof of the garage?
**Twin paradox**

Twin paradox is one of the famous paradox in the TR. One of the twin – astronomer – flies on the space mission to the planet Z and returns after 20 years. His twin brother who stayed on the Earth is 40 years older when they met. How it is possible if the motion is relative? We can say, that brother on the Earth leaves the twin (astronomer) and returns back at the same speed.

Explanation of this paradox we can show in the LD. We have to divide the mission in two parts, the fly to the planet Z and the fly back, because the orientation of velocities changes. The first part of the trip is shown in the Fig. 17. For better understanding we colour background of the text same color as the frame. Red colour is the Earth frame and blue colour is for astronomer frame.

The twin on the Earth is in the rest in red frame and for his calendar the brother astronomer arrived to the planet Z in year 2420, twenty years after start. The twin on the Earth observes the date in the brother frame and due the time dilation there is only year 2410. Due the length contraction is distance between Earth and planet Z reduces to half and for astronomer it really takes only 10 years. When the astronomer in year 2410 and just before the orbit looks at the Earth, there is only year 2405. The situation is constructed in LD in the figure 17.

After orbited the planet Z the astronomer is flying back to the Earth with the same speed. We need to construct the new diagram because the astronomer changed direction of speed. The astronomer’s new frame is $\mathcal{K}''$. For illustrative we keep the time axis of the astronomer and space axis of twin on the Earth unchanged. The space axis of astronomer and time axis of the twin on the Earth have to be changed because of rule of the LD (Fig. 18).
For the twin on the Earth the astronomer flies back to the Earth in year 2420. For the astronomer in his frame it was in year 2410 (Fig. 18: axis $x''$). For him on the Earth is year changed to 2435 as it is shown in LD which is due changing frame from $\mathcal{K}'$ to $\mathcal{K}''$. The astronomer is still close to planet Z but changing frame due the change of simultaneous events. (This effect does not happen for twin on the Earth because he does not change the frame.) The fly back takes for astronomer 10 years but from his point of view takes only five years on the Earth. When he arrives on the Earth, there is the year 2440 but for astronomer is year 2420. For twin on the Earth passed 20 years for astronomer’s fly back. The whole trip took 40 years for twin on the Earth and for astronomer 20 years.

When they meet the difference between their ages is actually 20 years. The reason why it happens is that the astronomer changes the frame several times but the twin on the Earth never.

**Detonator paradox**

We have two components, one is the shape of letter U and other is shape of letter T. Let U and T components are made so that T part can tuck into U part, but it cannot reach a stem at the bottom of the U part. We put detonator inside the U part but if we put together U and T components, the stem does not activate the detonator (Fig. 19 left).

What does happen when T part approaches very fast the U part?

From the black frame of the T part the U part has to be shorten because of length contraction and the detonator is going to be activated and TNT explodes (Fig. 19 right 1.). From the red frame of the
U part the T part has to be shorten because of length contraction and the detonator is not going to be activated (Fig. 19 right 2.). What will happen?

Construct this situation in the LD (fig. 20). The T part is in the black frame in the rest and U part is in the red frame in the rest. At first we have to find the world-lines of the point which defines the length of components. Let the beginning of the U part to be the point A and the end is point B. For the T part the beginning is the point C and end is the point D (Fig. 20). The world-lines of the point A and B are parallel with axis $ct'$ because of red frame. The world-lines of the point C and D have to be parallel with $ct$, because of black frame.

The intersection of world-lines defines important events. Event 1 shows that point B is passing the point C. It means that the T part starts to tuck to the U part. Event 2 shows the situation that the point B hits the point C. In this event the U part should stop the motion. The event 3 shows the situation when the point A hits the point C. It means that the detonator will be activated and TNT will explode.

It is important in which frame the detonator is. As we can see the order of events is different for black frame as well as for the red frame. In the black frame for T part the event 3 happens first and after event 2. In the red frame for U part it happens vice-versa. The first happens event 2 and after event 3.

The maximal possible speed is speed of light. The information about event 2 that the points C and A should be stopped can’t be delivered earlier as the light which comes to the point C and A from event 2. In the event 4 this information arrives to point A. After in the event 5 arrives this information to the point C. As we can see the event 3 happens earlier than event 4 and 5. The detonator will be activated and the TNT bomb will explode.

![Fig. 20 UT paradox](image-url)
Task 1: Try to find maximum speed between frames using Geogebra that the detonator will not be activated. The length of the U part is 10 units and T part is 9.5 units.

Flickering bulb paradox
We have here another paradox that shows the difference observation between different observers. We have the circuit with bulb and slider which are shown in the Fig. 21 top. The slider with shape H can move on two parallel wires. On one wire is the branch (from A to B), which has the same length as slider H (point C and D). Connection between C and D of the slider H is not conductive.

![Fig. 21 Flickering bulb paradox](image)

The slider moves very fast. Slider is in the red frame and branch with bulb is in the black frame. In the black frame (bulb) due to the contraction of length that the slider is shorter and the circuit will be interrupted when the slider passes the branch (Fig. 21 middle). In the red frame (slider) the branch is shorter than slider and the circuit will not be interrupted (Fig. 21 bottom).

The current flow in the circuit determines whether a bulb is lit or not. When the current flows through the bulb then it also has to flow through the slider H. We can say that the part C of slider is the same as the D part then we consider that through the part C and D flows the same current $I_2$.

To construct this situation in LD (Fig. 22). We have to find world-lines of the parts C and D in the red frame (slider) and points A and B in the black frame (bulb). The intersection of this world-lines defines important events. Even 1 characterizes interruption part D on the slider and point A. It means that through the part D the current stops to flow. However the information can travel only by the speed of light. After it arrives to part C the current really stops through the part D. This causes the growth of the current through the part C to value $I_4$.

---

4 We do not consider the time dependence of current increase because it can be regarded as almost immediate and it will not affect the essence of the paradox. The bulb for a short time will shine less intensively, but soon after the intensity of the radiation scattered back to its original value.
When the part C of the slider is passing the point A – event 3 – the circuit is going to be interrupted. As well as in the situation before the information can travel maximal speed of light. The current will be flowing until the signal from event 3 arrives to the bulb. The event 4 corresponds the reconnection of the circuit. This information will also travel maximum with speed of light.

From LD it can be seen clearly that the bulb in black frame did not work in the time of interval $c t_{34}$. Because the bulb is in the black frame so the observer in the red frame should see the same but the time interval when the bulb did not work is $c t'_{34}$ as is shown in LD (Fig. 22).

**Fig. 22** Flickering bulb paradox

Task 1: What will change if the slider moves in the opposite direction? How the LD will change?
2 Relativistic mass

In classical physics we know the term center of mass or barycenter. For two points on one line with masses of $m_1$ and $m_2$ is center of mass defined by this equation

$$m_1 r_1 = m_2 r_2.$$

We have two mass points $m_1$ and $m_2$, and they move apart in the way of axis $x$. For observer in black frame the position of the center of mass in time $cT$ this equation is valid

$$m_1(0)r_1 = m_2(v)r_2.$$ (1)

In this case mass of moving object 2 will depend on moving estate of this body. This is the same like the case with a clock and length of the rod. This situation is shown in LD in fig. 23. A mass point $m_1$ is in the rest in black frame, is world-line of mass point identical with axis $ct$. For mass point $m_2$, which is in the rest in red frame, is world-line of mass point identical with axis $ct'$. The movement of center of mass for this two bodies fits exactly of one world-line. World-line of center of mass is uniquely determined in LD.

We can use the same point of view for observer in red frame (Fig.24). For center of mass can be written

$$m_1(v)r'_1 = m_2(0)r'_2.$$ (2)

![Fig. 23](image1.png) The situation in the black frame

![Fig. 24](image2.png) The situation in the red frame

From Loedel diagrams in fig. 23 and 24 we can see, that due to relativity is simultaneous of events in black frame and in red frame the distance ratio of mass points 1 and 2 from the center of mass in different ratio

$$\frac{r_1}{r_2} \neq \frac{r'_1}{r'_2}.$$

In the black frame is the mass point 1 in the rest, but mass point 2 moves by speed $v$. In red frame is everything reversal. The ratio of distances between mass point and center of mass is not the same (but definition says it must be), we can see that the mass depends on velocity.
Therefore applies

\[ m_1(0) \neq m_1(v), \]
\[ m_2(0) \neq m_2(v). \]

We can base again from the definition of the center of mass, from which (thanks to the principle of relativity in each inertial system) equations (1) and (2) must apply

\[ \frac{m_1(0)r_1}{m_2(v)r_2} = 1 = \frac{m_1(v)r'_1}{m_2(0)r'_2}. \]

(3)

From resemblance between blue orthogonal triangles in Fig. 25 ensue, that

\[ \frac{r'_1}{r_1} = \frac{r_2}{r'_2} = \cos \alpha. \]

Fig. 25 Relativistic mass

Now arrange quantities from equation (3), that we will have mass on the left side and distance on the right. We will have

\[ \frac{m_1(v) m_2(v)}{m_1(0) m_2(0)} = \frac{r'_2 r_1}{r_2 r'_1} = \frac{1}{\cos^2 \alpha}. \]

Mass \( m(v) \) of body moving with velocity \( v \) apparently proportional of mass \( m(0) \) in the rest, so have to be general applies

\[ m(v) = m(0)f(v), \]

so

\[ \frac{m_1(v)}{m_1(0)} = \frac{m_2(v)}{m_2(0)} = f(v). \]
The \( f(v) \) is some function, which does not depend on the rest mass of bodies, but this function is the same for each body. Then have to apply

\[
f(v) = \frac{1}{\cos \alpha}
\]

Due to substitution we have

\[
m(v) = \frac{m(0)}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

and due to this we have one of special prediction of a special relativistic theory.

3 Doppler Effect

Light is an electromagnetic wave, so we can assign a wavelength and frequency of it. Considered the light signal in the LD and three consecutive zero values which determined the wavelength \( \lambda \). Let’s show the first and third timescale with zero value in red and black frame, corresponding to the period \( T \) respectively \( T' \) of waves in various systems in LD.

![Doppler Effect](image)

**Fig. 26** Doppler Effect

What is relation between periods registered by individual observers?

From green orthogonal triangle we get distance \( |SX| \)

\[
|SX| = cT \cos \alpha
\]

and also for the distance \( |SX| \) we can write

\[
|SX| = cT' + \lambda' \sin \alpha.
\]

Being a light signal moving at \( c \), then we get

\[
cT' = \lambda'.
\]
Due to comparison a substitution we get this equation

\[ T = T' \frac{1 + \sin \alpha}{\cos \alpha} = T \sqrt{\frac{(1 + \sin \alpha)^2}{\cos^2 \alpha}}. \]

After modifying and substituting basic trigonometric equations \( \sin^2 \alpha + \cos^2 \alpha = 1 \), \( \sin \alpha = \frac{v}{c} \) we get

\[ T = T' \frac{c + v}{\sqrt{c^2 - v^2}}, \]

\[ f = f' \frac{c - v}{\sqrt{c^2 + v^2}}. \]

This equation describes Doppler Effect, which can be seen at glowing in the fast-moving subjects.
References


Using silent videos in the teaching of mathematics
Introduction

GeoGebra, www.geogebra.org is dynamic mathematics software for teaching and learning mathematics at all school levels. It is open source and has been translated to many languages. In this chapter we describe some innovative uses of GeoGebra together with screencast technology. This activity was started by teachers in the Nordic and Baltic countries working with the Nordic GeoGebra Network, which is a network of eight Nordic and Baltic countries. We will describe the networks’ activities that aimed to facilitate collaboration and sharing among teachers and researchers, and focused on the use of ICT in the teaching and learning of mathematics. We will give a brief introduction to how the project started, explain the use of silent videos and give a link to many examples that have been tested by teachers in Iceland, Estonia, Lithuania, Sweden and Latvia.

Nordic GeoGebra Network

The software GeoGebra is used for teaching and learning mathematics at all school levels (Hohenwarter, Hohenwarter & Lavicza, 2009). It has an extensive user community, local GeoGebra Institutes in many countries, and an International GeoGebra Institute that maintains a website www.geogebra.org (Hohenwarter & Lavicza, 2007). In each of the Nordic and Baltic countries there have been efforts to promote the use of GeoGebra and other software in the teaching and learning of mathematics. Mathematics teachers from University, upper secondary school and lower secondary school founded the Nordic GeoGebra Network (NGGN) in 2010.

The aims of NGGN are:

- To create opportunities for mathematics teachers to learn from each other about the use of ICT/GeoGebra in the teaching of mathematics.
- To create possibilities for teachers and researchers across the Nordic and Baltic countries to collaborate on research projects regarding the use of ICT in the teaching and learning of mathematics.

The network runs the website http://nordic.geogebra.no/. It received grants from Nordplus for the years 2010 – 2013 and 2014 – 2016.

Activities

The main activity of NGGN is the organisation of a yearly conference that has been held in Iceland (2010), Lithuania (2011), Estonia (2012), Denmark (2013), Finland (2014), Sweden (2015), and Norway (2016). Our main effort is to encourage teachers to share their experiences with each other through workshops and talks. The Network members also give talks and workshops as needed. At the conferences we have plenary speakers that are researchers in Mathematics Education and regular teachers from secondary and primary
school as well as one plenary speaker from the GeoGebra Team. The conferences usually have 80 – 100 participants.

In order to encourage collaboration among teachers in our countries we started organising key-topic groups that would work on specific topics between the conferences. The first key-topic group was formed in 2013.

For each conference we plan some lectures related to that years’ key-topic, form a group (open to everyone interested) at the conference to work with the key-topic until the next conference, and schedule a group meeting between conferences for the group to prepare and organize their work. At the next conference the group gives a plenary lecture on its findings. A network member chairs the group but the people in the group decide themselves what they want to work on. The key topics chosen were: Learning mathematics through screencast technology and video (2013/2014), development and use of resources (2014/2015), teacher knowledge and classroom management in a technological environment (2015/2016), and assessment of mathematical competencies in a technological situation (2016).

**Screencasts**

The first key-topic group on “Learning mathematics through screencast technology and video” (SC group) was created in Denmark in 2013. It had 20 participants that after some brainstorming came up with a variety of ideas to work on. The SC group met in Iceland in the spring of 2014 for preparations, and gave a plenary lecture at the conference in Finland in 2014, [http://hylblog.edu.hel.fi/wpmu/ggylojarvi14/programme/](http://hylblog.edu.hel.fi/wpmu/ggylojarvi14/programme/). The members of the group were from Iceland, Sweden, Estonia, Latvia and Lithuania. Many of the people who joined the key-topic group at the conference dropped out but others replaced them since participants in each country managed to recruit many others who were interested in this particular topic.

Of the many ideas that this group came up with only one survived, namely the idea of using **silent screencasts** which we thought of as a tool that could be used in all our countries independent of our different languages. The screencasts were short videos without a commentary, showing a mathematical concept dynamically in GeoGebra.

**The work of the Screencast group**

After the Screencast group started focussing on the silent videos the idea of having students make commentaries to the silent videos emerged. The group made many silent videos on different mathematical concepts in the spring of 2014. They shared those videos via Dropbox, reviewed and discussed them. Eventually they agreed to use three of the videos for further testing, these three videos along with further six can be viewed at [http://www.geogebra.org/material/simple/id/98465](http://www.geogebra.org/material/simple/id/98465). The ones used by the Screencast group were: Triangle area and height (3), line symmetry (6) and point symmetry (7).

The videos were made by using the free software Screencast-o-matic to record some mathematical phenomena in GeoGebra. They are all quite short (1-2 minutes) and show no text or written explanations. The idea is that teachers can use these to see if or how much students understand from purely visual explanations.
The group tested their material in 2014. Mostly this was conducted in such a way that students worked in pairs, watched the silent videos as often as they liked, prepared their own commentary, and used screencast-o-matic to record it. A total of approximately 200 groups/pairs of students participated in the project with their teachers in Estonia, Iceland, Latvia, and Lithuania. One of the videos was also tested in Sweden in a pilot study to prepare the other tests.

The participating teachers are: Bjarnheiður Kristinsdóttir (Iceland), Rokas Tamosiunas and Marius Zakarevičius (Lithuania), Ilze France, Anete Zača and Evija Slokenberga (Latvia), and Sirje Pihlap and her teacher-training students (Estonia). The teachers that did the pilot study are: Marie Utterberg and Eva-Lena Cederman (Sweden). In addition to the pilot study, Marie Utterberg and Eva-Lena Cederman did some tests for their PhD projects where they asked pupils to use GeoGebra when solving mathematical problems and recorded their solutions process with screencast technology.

For practical reasons, the teachers in different countries implemented the experiment in slightly different ways. Some were not able to book or had no access to computer rooms and therefore had to ask their students to write about the screencasts instead of recording a screencast with their commentary.

**The teachers’ experience**

In general the teachers who took part in the experiments noticed that through the students’ commentaries they were able to gain insight into the students’ way of thinking. This insight was something that they would not have gained otherwise. They also experienced that working on this task stimulated discussions on mathematics among the students and the teachers could observe how they communicated and react on their discussions. This way they felt they were able to enhance the students’ motivation to think and learn.

The ready-made video with commentaries from a pair of students could be used by other students to learn. This fact was known to all the students beforehand and had the positive effect that they put more effort into making a good commentary.

For the most part the students enjoyed working this way, as it was different from anything they had tried before.

**Evaluating results**

To evaluate the conceptual understanding of the students one can consider using the SOLO-taxonomy model which distinguishes between pre-structural, uni-structural, multi-structural, relational and extended abstract understanding (Biggs & Collis, 1982; Biggs & Tang, 2007).
The Screencast group has so far done some preliminary evaluations of their students’ understanding using this taxonomy. The preliminary results are shown in Figures 2 and 3.

**Fig. 2** Evaluation of the understanding of pupils who did not know the mathematical concepts beforehand

**Fig. 3** Evaluation of the understanding of pupils who knew the mathematical concepts beforehand
Using silent videos for assessment – an example

Two teachers in Iceland, Vilhjálmur Sigurjónsson and Guðrún Angantýsdóttir have created two complete teaching units with material, exercises, projects, ideas for assessment etc. One of the units is for the equation of a line https://sites.google.com/site/jafnalinu/home and the other one for second degree functions: https://sites.google.com/site/fleygbogi/.

One of the assessment ideas is to use a silent video together with an evaluation sheet that a teacher can use or students can use for self-assessment. This silent video can be found at https://sites.google.com/site/jafnalinu/namsmat---hljodhlaust-myndband. Together with the video an evaluation sheet is given. A translation of the evaluation sheet is given below.

This material has been tested and received a positive response from the students that tried out the silent video.

**Evaluation sheet - translation**

**Equation of a line – understanding concepts**

**Making a commentary to a video**

Students work together in pairs. They watch a silent video on the equation of a line as often as they want to. They use some recording software, e.g. Screencastomatic.com, or use their phone to record their commentaries. The students are expected to make a commentary to the silent video and explain what goes on in the video. They put the video in the recording window, start the recording and play the video as they talk. It is most suitable if the students have decided beforehand what they want to say before they start the recording. After they finish the recording, they send it to their teacher or publish it on YouTube and send a link to their teacher via email.

It is recommended to advice the students to use the concepts they have been working on since it is not certain that they realize to do so.

On the next page there is an evaluation sheet for the teacher to use when evaluating the students work.

The silent video is here (http://youtu.be/UEP3f_oYrMs).
Evaluationsheet. Equation of a line - video

Name/names: _________________________________________ Point total (100):

### 3.1.1 y-intercept (30 points)

<table>
<thead>
<tr>
<th>The student knows that the coefficient ( b ) gives the intersection of the line and the ( y )-axis. He gives at least two examples e.g. ( y = 0 ) and ( y = 2 ).</th>
<th>A B C D E</th>
</tr>
</thead>
<tbody>
<tr>
<td>The student knows that a straight line with a defined slope intersects the ( y )-axis.</td>
<td></td>
</tr>
</tbody>
</table>

### 3.1.2 Slope of a line (20 points)

<table>
<thead>
<tr>
<th>The student gives examples on vertical and horizontal lines. He/she knows that when the slope is 0 the line is horizontal and when the slope is undefined the line is vertical.</th>
<th>A B C D E</th>
</tr>
</thead>
<tbody>
<tr>
<td>The student knows that a line has a slope but does not give an example of a horizontal or a vertical line.</td>
<td></td>
</tr>
</tbody>
</table>

### 3.1.3 Slope of a line (30 points)

<table>
<thead>
<tr>
<th>The student knows that the coefficient ( a ) gives the slope of the line. The student realizes that when the slope is positive the line goes upwards from the left to the right but when the slope is negative the line goes downwards.</th>
<th>A B C D E</th>
</tr>
</thead>
<tbody>
<tr>
<td>The student knows that a line has a slope but gives no examples on the slope of a line.</td>
<td></td>
</tr>
</tbody>
</table>

### 3.1.4 Slope of a line (10 points)

<table>
<thead>
<tr>
<th>The student gives some examples on different values of the slope e.g. ( a = 2 ) and ( a = -5 ).</th>
<th>A B C D E</th>
</tr>
</thead>
<tbody>
<tr>
<td>The student knows that a line has a slope but gives no examples on the slope of a line.</td>
<td></td>
</tr>
</tbody>
</table>

### 3.1.5 Slope of a line (10 points)

<table>
<thead>
<tr>
<th>The student realizes that when the slope is e.g. 2 this means that for each step in the direction of the positive ( x )-axis the line goes up by 2 steps is the direction of the positive ( y )-axis.</th>
<th>A B C D E</th>
</tr>
</thead>
<tbody>
<tr>
<td>The student knows that a line has a slope but gives no examples on the slope of a line.</td>
<td></td>
</tr>
</tbody>
</table>

Translated with permission from the authors, Vilhjálmur Þór Sigurjónsson and Guðrún Angantýsdóttir
Making a Screencast - some guidelines

The software Screencast-o-matic can be found at the website https://screencast-o-matic.com/home. There is a free version and a more sophisticated one that can be bought for a small fee. Another widely used screen-casting software is Jing, which can be found at the website www.jingproject.com, and for a full list with comparison one can check out Wikipedia: https://en.wikipedia.org/wiki/Comparison_of_screencasting_software

The first step is to get an idea and decide which mathematical concept one wants to show and if GeoGebra is used, make a worksheet with a construction that can be moved in some way or use the Navigation bar in GeoGebra which can be used to play through your construction. When using a slider it is recommended to use the arrow keys on the keyboard rather than using the mouse since the appearance of the mouse pointer can be disturbing for the viewer.

Once Screencast-o-matic or another screencast-software is on the computer it is very easy to start the program, which will then record anything on the screen within the rectangle that appears. The size of the rectangle can be adjusted as wanted.

It is important to make the video short (1-2 minutes) and preferably to include only one concept or at least try not to include too many concepts in one video. If the students feel that they need more time to explain the concept, they can replay the video or push the pause button while recording their screencast.

In case one wants to use the silent video to test conceptual understanding or introduce a new concept is also important to avoid giving any clues in the name or description of the video when uploading it to YouTube or a local server.

References


Modernization of Education Using Geographical Information Systems
Modernization of Education
Using Geographical Information Systems

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Introduction

Educational process is a system of three main components, which are in constant mutual interaction. These components are namely the teacher, the curriculum, and the learner. Nowadays it is a customary convention that the curriculum is prepared exclusively by the teacher who, using various teaching methods and organisational forms of instruction, exposes the curricular content to the learners. Afterwards, learners try to learn the curricular content and they are expected to be able to present their newly acquired knowledge in written and/or oral form. Finally, the teacher assesses the learners, especially to what extent they are able to reproduce the curriculum.

The above described principle is one of the reasons why learners build no close relationship with the curriculum. As a matter of fact, it leads to passive knowledge acquirement and focuses just on gaining knowledge at the expense of development of skills and attitudes. Knowledge is acquired in isolation, without any particular links to practice and to the content of other school subjects. Learners get no opportunities to influence the curriculum, they can see no possibilities of practical use of their knowledge, and thus, they are hardly motivated to try to comprehend the curricular content. Their only objective is to keep the knowledge in their minds till they present it to the teacher and get evaluated.

In order to change this customary system of education the instructional process must be subjected to effective modernization, touching all components of educational process. It is necessary to change the teachers’ approach and their role of lecturers must be changed to role of educational process managers. The curriculum needs to be made more attractive, and both material and non-material teaching means needs to be innovated.

Geographical information systems (GIS) have a strong potential to be an appropriate tool for modernization of education in a pedagogue’s hands. GISs allow theory to be linked to practice, provide learners with a lot of space for activity, and are an excellent means of emphasizing the interdisciplinary relationships.
Definition of GIS

Just like the field of geography, the term Geographic Information System (GIS) is hard to define. This is because defining and understanding the term of GIS can be approached in three different levels described below.

1. **GIS as software.** On this level term GIS refers to programme products which serve for data management and data analysis. A typical show of someone’s exclusive personal identification with this perspective is preference of certain software tools and the notion that a concrete product developed by a concrete company is the one and only real GIS. Let us agree with Rapant (2002), Tuček (1998), and several other authors who believe that using term GIS only in such a narrow meaning is wrong. It may be only stated if certain software is appropriate for building GIS or not. In terms of the software perspective, three main types of GISs can be distinguished: desktop GIS (software QGIS, ArcGIS, GRASS, and the like), GIS server (e. g. services WMS), and mobile GIS (GPS).

2. **GIS as concrete applications.** Perceiving GISs as applications which are provided by them is present in many definitions of GIS. Such an example is also a well-known and frequently cited definition of GIS according to Burrough (1986), stating that GISs represent “a powerful set of tools for collecting, storing, retrieving at will, transforming and displaying spatial data”. The most frequently used applications of GISs are the following:
   - **mapping locations:** maps creation through automated mapping, data capture, and surveying analysis tools.
   - **mapping quantities:** people map quantities, like where the most and least are, to find places that meet their criteria and take action, or to see the relationships between places.
   - **mapping densities:** While you can see concentrations by simply mapping the locations of features, in areas with many features it may be difficult to see which areas have a higher concentration than others. A density map lets you measure the number of features using a uniform areal unit, such as acres or square miles, so you can clearly see the distribution.
   - **finding distances:** find out what is occurring within a set distance of a feature.
   - **mapping and monitoring change:** GIS can be used to map the change in an area to anticipate future conditions, decide on a course of action, or to evaluate the results of an action or policy (Young, 2009).

3. **GIS from the perspective of information technologies.** This approach represents the most general and broadest understanding of term GIS. From this perspective GIS can be defined as “an information technology which stores, analyses and displays both spatial and non-spatial data” (Parker, 1988). In fact, this approach represents a complex view on GIS as hardware, software and organisational environment in which the particular applications of GIS originate and take place. This environment consists of the following structural components (Jedlička, Brehošťovský, 2003):
   - **Hardware** – includes computers, computer networks, input and output devices (geodetic devices, terrestrial and cosmic segment of the global positioning system GPS – digitisers, plotters, scanners etc.).
   - **Software** – The software for work with geographic data is often modular. The base of the system is the kernel which includes standard functions for work with geodata and programme extensions (modules) for specialized works (e. g. processing of photogrammetric
photos and photos of distance survey of the Earth, network analyses, spatial and statistical analyses, 3D projections, creation of cartographic outputs).

- **Data** – Data are the most important component of GISs (as much as 90% of financial expenses on GIS functioning consist of means of data acquisition and updates).
- **People** – GIS users – programmers, GIS specialists (analysts), end users.
- **Methods** – methods of using a certain GIS, its fusion with an existing information system of the using company (from the practical point of view very intricate and demanding part).

Based on the broader sense of GIS in all the three dimensions, GIS can be defined as follows: “GIS is a system of hardware, software, data, people, organizations, and institutional arrangements for collecting, storing, analyzing, and disseminating information about areas of the earth” (Dueker and Kjerne 1989).

**GIS – a tool in hands of a pedagogue**

Capacity of GIS to retrieve, store, analyze, model and map large areas with huge volumes of spatial data has led to an extraordinary proliferation of applications.

Geographic information systems are now used for land use planning, ecosystems modelling, landscape planning and assessment, transportation and infrastructure modelling, market analysis, visual impact analysis, watershed analysis, facility management, real estate analysis, and many other applications (Grežo, Jakab, 2014).

The progressive development of GISs, development of simplified software for wide variety of users, emerging applications usable in professional practice as well as in everyday life, have naturally led to increased interest in this issue in commercial sphere, too. GIS tools are gradually appearing in educational process in primary and secondary schools where they can play an especially important role in efforts to modernize the school instruction and to motivate and activate learners.

Tools of geographic information systems can be helpful for teachers when:

- teaching subjects whose educational content has a spatial character,
- collecting, processing and presenting spatial data,
- motivating learners to active knowledge acquirement in terms of inquiry-based learning,
- making efforts to link theory to practice during instructional process,
- working with gifted learners,
- building interdisciplinary relationships,
- educating through project-based learning,
- teaching cross-sectional issues, such as Project Creation and Presentation Skills, and Environmental Education.

The broad employment of GISs in sciences and research and the appropriateness of their incorporation in educational process are in correspondence with the interdisciplinary approach on which GISs are based (Fig.1).
Users utilize their informatics knowledge as they work with tools based on database which in this case has a spatial character. Users can use software, the internet and map servers for collecting and sharing data, and computer graphics for data visualisation. Data included in GIS, so called geospatial data, consist of spatial and thematic component. The spatial component, so called geometry, serves for positioning of data with the use of coordinate systems and mutual relationships between the particular features of geospatial data. Defining the geometry of geospatial data is based on mathematical principles and follows geographic and cartographic rules. Depending on their application focus, the particular analyses utilize mathematical models, programme procedures, remote sensing system, graphs of mathematical functions, graph theory, networks, and the like.

For educational objectives there are many web, mobile and desktop applications available. Among desktop application there is a wide choice from both commercial and non-commercial software.

For utilization of GISs in educational process in primary and secondary schools we prefer non-commercial freely available software, so called Open Source Geographic Information System, which bring about several advantages (Grežo, Jakab, 2014):

- There is no need to have a budget for this Professional software. Its installation, use and updating is free of charge.
- Openness of the software allows easy data exports and their transformation from one software to another, which allows combination of several different software in tasks solving.
- The main advantage is that teachers and learners can install them freely in their computers and work in professional GIS software is independent from availability of a computer room.
• By offering certain rights and possible availability of the source code of software the development of GIS tools may take place also outside large companies. This fact can be a strong motivating factor for students who can develop new tools or transform existing ones.

• Graduates who come to professional practice are not only professionally competent, but they know the possibilities and advantages of Open Source software as an alternative to the use of professional GIS software, and are able to use them directly.

• Use of Open Source GIS software is a convenient alternative for life-long education in the field of GIS. The use of GIS software in education process at lower and higher secondary level is a modern and efficient means of teaching science applicable in the teaching practice of the current educational system. This results in an increased interest in this area, and the option of education using Open Source offers in addition to trained teachers also a free tool for schools.

• There is available a variety of materials and manuals.

From real world to GIS environment – data models

Transforming real objects and processes to GIS environment is not an easy task, for reality is irregular and constantly changing, therefore the perception of reality strongly depends on the observer. For instance, a geodesist perceives a road as two edges which are the subject of the measurements, whereas for a road maintenance worker the same road represents a tar surface whose quality has to be sufficient for unfettered traffic, and a car driver perceives the road as a motorway where he can drive at a certain speed.

From several perspectives GIS represents a simplified view on the real world. If we simplify the real world, while intentionally keeping some parts of the real world so that the final representation has certain required properties common with the real world, then the result of this process can be considered a model. Models occur in various forms. One of the forms is a database which can contain a considerable amount of data and also provides several functions for operations with the saved data. In GIS environment one of the most frequent models is a map. A map is a simplified representation of selected parts of the real world (Huisman, de By, 2009).

A map in terms of its content is a spatial database of spatial objects, drawn on paper or saved in digital form.

Data modelling is a process of abstraction when substantial elements of the real world are emphasized and the insubstantial elements are eliminated, yet, with regard to the specific objective(s) of the modelling process.

Data modelling can be defined in the context of geographic information systems (GIS) as occurring whenever operations of the GIS attempt to emulate processes in the real world, at one point in time or over an extended period (Goodchild, 2005).
The result of data modelling is a **data model** which is a formal representation of the real world that provides a standard structure for storage, sharing and exchange of data independent of the software environment and programming languages (Kumar, 2010).

The process of data modelling consists of four main levels (Molenaar, 1998; Peuguet, 1984), which are illustrated in Fig. 2.

1. The first level is the **reality** itself – the real structure of phenomena perceived by the user. It consists of phenomena in their real form, including all its properties which may and do not have to be important for given application (Voženílek, 1998).

2. The second level is represented by the **conceptual model** which is the initial stage of abstraction and comprises only those components of reality that are considered essential for given (specific) application. An apt example of the conceptual model actualisation is a map, as it comprises only those features of phenomena which were chosen by the cartographer for representation and all the remaining aspects of reality are omitted. The required properties of reality are still well interpretable from the map or can be easily reconstructed. Just like a cartographer creating a map has to decide what symbols should be used for various phenomena, at the stage of conceptual modelling it has to be decided what data representation will be used (either vector or raster) and what map layers will be created for given topic. The conceptual model is often referred to as data model (Brimicombe, 2010).

3. The third level is the logical model, often referred to as **data structure**. It represents the future abstraction of the conceptual model into records, fields and matrices (Longley et al, 2005). It determines the way geoelements from the conceptual model will be inserted and displayed in the database, controlled and recorded in the software code (Brimicombe, 2010). The logical model may be generally interpreted as reality only by means of software (data visualisation in GIS environment).

4. The fourth level is the physical model, so called **file structure**. This is the final abstraction which represents how data are physically saved to the computer hardware or other media in form of bits and bytes (Brimicombe, 2010).
Understanding the particular levels of data modelling is the essential pre-condition for successful use of GIS in educational process. Also, the particular levels bring logical order in creation of data models – geospatial data for GIS.

**Project-based learning by means of GIS**

In order to make GIS tools and applications an effective tool in hands of pedagogues close attention has to be paid to their implementation in the educational process. Their greatest potential is seen in their use for learners’ projects in terms of project-based teaching.

Project-based learning (PBL) is an instructional model that is based in the constructivist approach to learning, which entails the construction of knowledge with multiple
perspectives, within a social activity, and allows for self-awareness of learning and knowing while being context dependent (Duffy & Cunningham, 1996).

Thomas (2000) sets five criteria for PBL:

1. **PBL projects are central, not peripheral to the curriculum.** According to this defined feature, projects are the curriculum. In PBL, the project is the central teaching strategy; students encounter and learn the central concepts of the discipline via the project.

2. **PBL projects are focused on questions or problems that "drive" students to encounter (and struggle with) the central concepts and principles of a discipline.** This criterion is a subtle one. The definition of the project (for students) must "be crafted in order to make a connection between activities and the underlying conceptual knowledge that one might hope to foster (Barron et al., 1998)

3. **Projects involve students in a constructive investigation.** An investigation is a goal-directed process that involves inquiry, knowledge building, and resolution. Investigations may be design, decision-making, problem-finding, problem-solving, discovery, or model-building processes. But, in order to be considered as a PBL project, the central activities of the project must involve the transformation and construction of knowledge (by definition: new understandings, new skills) on the part of students (Bereiter & Scardamalia, 1999).

4. **Projects are student-driven to some significant degree.** PBL projects are not, in the main, teacher-led, scripted, or packaged. Laboratory exercises and instructional booklets are not examples of PBL, even if they are problem-focused and central to the curriculum. PBL projects do not end up at a predetermined outcome or take predetermined paths. PBL projects incorporate a good deal more student autonomy, choice, unsupervised work time, and responsibility than traditional instruction and traditional projects.

5. **Projects are realistic, not school-like.** Projects embody characteristics that give them a feeling of authenticity to students. These characteristics can include the topic, the tasks, the roles that students play, the context within which the work of the project is carried out, the collaborators who work with students on the project, the products that are produced, the audience for the project's products, or the criteria by which the products or performances are judged. Gordon (1998) makes the distinction between academic challenges, scenario challenges, and real-life challenges. PBL incorporates real-life challenges where the focus is on authentic (not simulated) problems or questions and where solutions have the potential to be implemented.

For effective implementation of GIS in the instruction through PBL it is recommended to follow the below stated directions.
Setting the educational goals

In efforts to implement GIS in the instructional process through project-based learning it is necessary to focus on both components of the instruction – its content and the educational process itself. Therefore, the first step to successful realisation of PBL is careful and thoroughly elaborated planning. Planning a programme for specific target group of learners usually starts with setting the educational goals.

A specific of PBL and GIS is their interdisciplinary character, which should be reflected in the educational goals. That is why teachers of various subjects should cooperate while planning PBL lessons.

Properly set educational goals are the foundation stone of PBL planning. Therefore, it is very important for teachers to consider several fundamental questions already in the initial stage of planning:

- What should be the educational content of the planned lesson(s)?
- What will be the topics of learners’ projects? What will be their outcomes?
- What school subjects or curricular issues should be included and in what extent?
- What key competences of learners should be improved?
- What existing knowledge and skills of learners will be supportive?
- What cross-sectional issues can be covered in the curriculum within PBL instruction?
- Is the time spent on PBL adequate to the knowledge and skills which should be acquired by learners during PBL instruction?

If educational goals are not set properly, the time and space devoted to PBL might happen to be used ineffectively, and efforts and energy spared by both teachers and learners might happen to be inadequate to the project results.

Topics for learners’ projects

Topics for projects of learners are devolved on the educational goals. The choice of the right topic is one of the crucial points in planning PBL lesson. Learners may have much negative experience with receiving information in which they can see no applicability and practical benefits for their future lives. Therefore, when choosing topic for projects it is very important to let learners know what the purpose of the activity is, what the expected results of their effort are, and what the whole point of it is.

GIS are suitable for variety of topics and for various extent of elaboration. Very effective seem to be PBL incorporating empirical discovering when group of learners role-play scientific teams whose objective is to conduct their own scientific research and then present their results and discoveries. There are plentiful themes open to effective use of GIS within such design of instruction:

- monitoring selected plant and animal species (e.g. invasive plants, curative herbs, aquatic and soil invertebrates);
- monitoring the quality of environment (e.g. water, soil, air);
proposing motions based on the current state analysis (e.g. designing educational trails, information brochures, advertisement shots, eco-pedagogical sites, children playgrounds, park revitalisation);

• terrain measurements (e.g. creation of maps by means of geodetic measuring principles);

• evaluation of landscape changes in time (e.g. comparisons of usage of certain area at present and in the past).

Learners may come up with proposals of topics for their projects as well. Yet, it is recommended that before telling learners the project topics the teacher subject the topics to thorough analysis, especially in terms of their content and process appropriateness, and time and safety perspective. When choosing topics for projects based on empirical discovering, the teacher must know the issue well, as well as the methods which should be used by learners during the project work, and learners should be provided with all the necessary means and tools required by the project objectives.

One of the pillars of PBL is the learners’ interest in the project topics. Teachers should therefore adjust the instructional process in such way that the overall atmosphere elicits learners’ inquiry, their will to search for answers, and their internal interest in the unknown. PBL incorporating GIS can be, thus, a rich source of stimuli increasing learners’ intrinsic motivation to learn.

Openness of project-based learning

Since learners and their individual thinking, interests and needs are so diverse, PBL should be organized with regard to the individualities of learners. The topic of the project should offer learners certain amount of freedom in decisions, which makes it their personal matter. Freedom in decision means that the teacher gives the learners a free hand in particular steps of the project work, or suggests several alternatives from which learners can choose by their own judgement. The opportunity to make a choice offered to learners has a strong motivational and activating potential. If learners choose by themselves the location where they will take a sample from, they will probably have no special comments on the accessibility of the place. If learners choose the form of their project results presentation, they are not likely to feel limited in their creative activities. If they choose the research sample, the number of measurements, or if they specify their research focus themselves, they will try to prove their decisions right by seeking to elaborate their project successfully. The open character of PBL can be beneficial for positive development of learners’ divergent thinking as well as their autonomy.

Emphasizing the interdisciplinary relationships

Another crucial factor influencing the efficacy of PBL is covering topics of several school subjects. A programme based on interdisciplinary relationships results in systematization of learners’ acquired knowledge, interconnects their knowledge and skills in several subjects and makes them able to look at studied phenomenon from various perspectives and take various approaches.

The interdisciplinary approach is just desired when using GIS in school instruction. For instance, mapping illegal waste dumps is in terms of content related to geography as well as civics. When processed by means of GIS, it is also related to computer sciences. If GIS serves for calculation of the
Projects of learners based on interdisciplinary relationships require active communication of teachers and designing their common instructional strategy. The strategy consists of:

- division of tasks related to preparation and realisation of PBL;
- preparation of time plan – planning tasks in terms of time demands;
- division of curriculum among particular school subjects, i.e. which subject provides learners with what information necessary for realisation of the projects;
- adjustment of the curriculum to the demands of the project topics.

**Linking the project with the surrounding world**

PBL may give more room for linking the instructional process with other school activities and with its broader surroundings. The links can be approached in two dimensions:

- Linking the topic with surroundings. For instance, at school level, when school life or school environment issues are being treated. The topic can be also linked at different levels, e.g. relating it to local issues, problems in the village, town, suburb etc.
- Also activities related to the project elaboration can be linked with town or village events.
- Links can be made in terms of cooperation of teachers and learners with professional organisations performing in the region by involving parents and local citizens in treatment of the issue.
- Possibilities of linking the projects with the surroundings are also provided by the learners’ final presentation of results. A kind of conference can be organized for learners, or a workshop, where also guests can be invited, such as the school management board, parents, local authorities, media etc. Various information boards can be prepared, an exposition can be organized in or out of school (galleries, leisure time centres), a website can be designed or a short movie can be shot and published.
Surveyors – a project for learners

The proposed project Surveyors makes use of GIS tools and the principle of data modelling in creation of a large scale map of the current state of selected piece of land (e. g. the school yard, a part of the town park or a housing estate). The project is intended for learners at the age of 13 and above. The extent of the project is not easily pre-estimated as it is influenced by the size of the area, the number of mapped objects, the chosen method, the depth of elaboration, and the visualisation of results.

The project is based on interdisciplinary relationships, and is directly related to mathematics, geography and computer sciences.

The wording of instruction is short and clear, but there are several possible ways of solution and each of them offers certain amount of openness, in accord with the PBL principles explained in the previous paragraphs. The openness can help develop learners’ divergent thinking, creativity and autonomy.

The initial stage of solving the task from the perspective of GIS use is spatial data collection. It seems reasonable to locale the objects with the use of GPS. Commercially used GPS tools work at too much inaccuracy which causes very high distortion in creation of a large scale map of a small-sized area. Therefore, it is necessary to apply geodetic positioning of the particular points. Purchase of geodetic devices is financially demanding, but applying the geodetic principles of positioning is possible in school conditions.

Position of the objects placed in the area is measured from measuring points using angles and distances measurement. Final proposals can be presented on the map together with already existing objects – pathways, trees, benches etc. The activity is equally time-consuming which allows for two groups of pupils, one for making proposal and the other one for drawing the map. In this case we get a group to work with differentiated tasks. This activity consists of these steps:

1. Recognition of interest area and existing objects.
2. Establishment of the conceptual model – choice of those parts of reality (those objects) which are considered important by the learners. A good way of representation of the conceptual model is to draw a draft map containing only those objects and those properties of phenomena which were chosen for representation of reality. All other aspects of reality are omitted.
3. Choosing the map scale. Indicating the northerly direction. Establishing the measuring points. Points can be measured by GPS devices, which can be further used for localisation of the measured objects into one of the existing GIS coordinate systems.
4. Measuring the size of the objects and their conversion to the scale.
5. Determine the position of objects in view to the measuring points. Pupils can choose the surveying technique: measuring angles, measuring distances or combination of measuring angles and distances. The third one is called Tachymetry.
6. In case the area at issue has significant inclination and the ground plans of objects measured at the inclination would be distorted in the map, it is possible to measure the inclination and use trigonometric functions in order to determine the real lengths.
7. Drawing objects in the map to scale. Creating the map legend and directional rose. For map drawing, pupils can also use drawing software, such as GeoGebra.

8. The map needs to be uploaded to GIS software environment so that data structure and file structure can be created within the data modelling process. In case learners made an analogue map in paper form, it needs to be scanned and then with the use of tools for geo-referencing it needs to be put into its place in the selected coordinate system. Geo-referencing involves assigning real world coordinates to a number of reference points on the image. This way a raster layer is formed which can be digitalized in vector format in GIS environment. In case the map was drawn with the use of software, it needs to be saved in svg format. Usually, format svg is not directly supported by GIS software. Therefore, it is necessary to use extension plugins which enable this, or help transform svg to dxf format accepted by GIS. For this purpose freely available software Inkscape can be used. Finally, we get a vector map layer whose constituent parts (features) have accurately defined geometry describing their shape and position of objects. Particular objects still do not have defined the attribute component.

9. Another step is adding attributes to particular features. This can be performed by directly inserting data by typing on keyboard (e.g. name of the object, its height, description, in case of woody plants their species and age, etc.), or by calculating with the use of GIS functions (e.g. the area and perimeter of objects, length of linear dimensions etc.). The result of this is the completed definition of the geometric and attribute component of all features.

10. The last step is visualization and cartographic presentation of geospatial data in form of a map. The advantage of using GIS is that they can process, edit, analyze and express in the visualisation both the spatial and the attribute component. If necessary, other features can be added, e.g. if learners are asked to propose new objects and phenomena in the map of current state. The shape of final features can be edited and their colour can be changed. Using means of cartographic presentation allows express information contained in the attribute component in form of a cartogram or cartodiagram. By adding the legend, north arrow and scale the final map can be created and exported in the selected graphic format (jpg, png, tif etc.).

The map of current state created in GIS environment represents a valuable product which can be used for other learners’ projects. Using analytical tools, the map of current state can be compared with historical maps, and the observed changes can be described. This way another school subject, history, can be covered in the interdisciplinary relationships. Other themes for use of the created map of current state can be as follows:

- monitoring the quality of water, soil, or air conducted in several places, related to physical and chemical properties measurement,
- proposing of an educational trail in selected area and positioning if the educational boards,
- determining the optimum position of objects in an area (e.g. a pond in the schoolyard),
- mapping selected plant and animal species (e.g. woody plant in the schoolyard, invasive plants, curative herbs, aquatic or soil invertebrates),
- proposing functional use of selected area (e.g. detailed proposal for building a playground at a grassed piece of land).

All the examples are based on interdisciplinary relationships, is linked to practice and offers open tasks for learners at secondary educational level.

In period 2011-2012 among other projects realized within a children summer camp in Recreation Resort Jedlínky children mapped the current state of selected area and designed new functional use of
the area. In both cases 16 children attended the activity, while half of them mapped the current state and created the map (surveyors), and the other half proposed new objects for the selected area (architects). Children were asked to solve the issue within one day, and the groups worked simultaneously.

At the beginning the group of surveyors obtained all information they needed for drawing the map of current state. They were familiarized with measuring methods and techniques. Subsequently, under supervision of the teacher, children started solving the task following the above described 10 steps. They started with demarcation of the mapped area and choice of objects and phenomena to be recorded in the map. Next, they determined the measuring points and decided for methods of measuring and positioning of the objects. The measured values (distances and angles between the measuring points and objects) were directly put in GeoGebra, where the primal representation of the area in a scale was developed. The complete data were then transported to GIS environment, particular objects were labelled, i.e. their attribute component was added. At the same time GIS served for final map visualization.

Design of the children playground was performed by the group of ‘Architects’, and the process consisted of the following steps:

1. **Getting to know the area of interest:** Architects had chosen to get familiar with the area for their first step. During its realization they focused on acquirement of information about the object of interest and its immediate surroundings. Main source of their information was not only observation but also conversation with the resort owner and study of available materials about the area.

2. **Evaluation of current condition:** After acquiring enough information pupils tried to evaluate current condition of the object and its services through brainstorming and discussion.

3. **Definition of prior measurement:** Critical assessments enable pupils to find solutions, identify weak points of the objects and consequent suggestions for improvement. Group of architects came up with an idea and proposal for a playground for children as there is not any in the resort. This would be appreciated by families and schools alike, being frequent visitors to the area.

4. **Forming the criteria for proposed changes:** Before elements of the playground can be chosen pupils have to set basic criteria to which they will refer when choosing play items. The following criteria were chosen: adequate size, reasonable price, variety of use, harmony with surroundings, safety, longevity.

5. **Selection of the areas for new functional use:** Base element influencing construction factor is allocation. Pupils found a suitable place for the playground near the swimming pool, 200 square meters of green area.

6. **Selecting particular components:** Using printed and online catalogues of garden architecture pupils chose a variety of swings, slides, play pits etc. according to their suggested criteria and place possibilities. The result of this is a narrow selection of components suitable for and agreed by both groups.

7. **Making the budget:** This step was realized along with the previous steps. Because price was an important factor when choosing components, it was inevitable to calculate and combine
individual proposals so that budget would not be exceeded. For the potential proposal the budget was made.

8. Registration of proposals: The final step is completing the map prepared by geodesist group including the changes proposed by architects group. This way a map of "what could be" is made by pupils.

In conclusion of the suggested program a presentation of pupils’ outcomes was organized. Pupils prepared presentations and described the process of mapping and creation of their proposals. They introduced the final product of their effort "A MAP OF THE FUTURE". Involvement and enthusiasm of pupils during the presentation, pride in their own results, disclosed experiences from implementation of the results, and actual quality of the resulting maps are positive feedback for us and reflect the efficacy of our teaching efforts.
References


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